EDITOR'S INTRODUCTION

When I was a young student in California, Lou Harrison suggested that I send one of my first pieces, *Piano Study #5 (for JPR)* to a Dr. Chalmers, who might publish it in his journal *Xenharmonikon*. Flattered and fascinated, I did, and John did, and thus began what is now my twenty year friendship with this polyglot fungus researcher tuning guru science fiction devotee and general everything expert.

Lou first showed me the box of papers, already called *Divisions of the Tetrachord*, in 1975. I liked the idea of this grand, obsessive project, and felt that it needed to be available in a way that was, like John himself, out of the ordinary. When Jody Diamond, Alexis Alrich, and I founded Frog Peak Music (A Composers' Collective) in the early 80s, *Divisions* (along with Tenney's then unpublished *Meta + Hodos*) was in my mind as one of the publishing collective's main reasons for existing, and for calling itself a publisher of "speculative theory."

The publication of this book has been a long and arduous process. Revised manuscripts traveled with me from California to Java and Sumatra (John requested we bring him a sample of the local fungi), and finally to our new home in New Hampshire. The process of writing, editing, and publishing it has taken nearly fifteen years, and spanned various writing technologies. (When John first started using a word processor, and for the first time his many correspondents could actually read his long complicated letters, my wife and I were a bit sad—we had enjoyed reading his completely illegible writing aloud as a kind of sound poetry).

Many people have contributed to the publication of this book, all volunteering their valuable time. David Doty (editor of *1/1, The Journal of the Just Intonation Network*) and Daniel J. Wolf (who took over publication of *Xenharmonikon* for several issues in the 1980s) both made a tremendous editorial contribution to style and content. Jarrad Powell, Joel Mandelbaum, David Rothenberg (especially for chapter five) and Jody Diamond made valuable suggestions. Lauren Pratt, who is to copy editing what John Chalmers is to tetrachords, saw countless errors that were not there until she pointed them out. Carter Scholz, the one person I know who can give John Chalmers a run for his money in the area of polymathematics, began as the book's designer, and by virtue of his immeasurable contributions, became its co-editor.

John Chalmers's *Divisions of the Tetrachord* is a fanatic work. It is not a book that everyone will read or understand. It is a book that needs to exist.

LARRY POLANSKY
Lebanon, New Hampshire 1992

FOREWORD

NEARLY TWENTY YEARS AGO John Chalmers and I had a number of very fruitful conversations. Well acquainted with the work of Harry Partch and also of younger musical theoreticians, Erv Wilson among them, John brought an immense amount of historical and scientific knowledge to our happy meetings. In turn, William Colvig and I brought the substance of professional musical life and the building of musical instruments.

At that time I had rhapsodic plans for a "Mode Room," possibly for UNESCO, in which would be assembled some great world-book of notated modes, their preferred tunings and both ethnic and geographic provenance, along with such history of them as we might have. I had supposed a roomful of drawers, each holding an octave metallophone of a mode, and somewhere a harp or psaltery of some further octaves' compass on which one might try out wider musical beauties of the mode under study. I even wrote out such a proposal in Esperanto and distributed it in an international ethnomusicology conference in Tokyo in 1961.

However, a little later Mr. Colvig began to build extremely accurate monochords on which we could study anything at all, and we rushed, in a kind of ecstasy, to try everything at once. Bill and I designed and built a "transfer harp," wirestrung and with two tuning systems, both gross and fine. Although innocently and quickly designed and built, its form, we discovered, is that of what the Chinese call a "standing harp"— the plate is parallel to the strings. We already owned a Lyon and Healey troubador harp, and, with these and with the addition of one or two other incidental

instruments, a bowed psaltery, drones, and small percussion, Richard Dee and I in one rapturous weekend tuned and recorded improvisations in a fair number of modes from planetary history, especially from the classical civilizations and Islam.

A little later, our friend Larry London, a professional clarinetist with wide intellectual interests and a composer of wide-ranging inquiry, made two improved versions of our original "transfer harp" and he actually revived what literature tells us is the way Irish bards played their own wirestrung harps, stopping off strings as you go. He has composed and plays a beautiful repertory of pieces and suites (each in a single mode) for his harps. I continue to want to hear him in some handsome small marble hall that reminds of Alexandria, Athens, or Rome.

Thus, the "Mode Room," about which I am still asked, turned into anyone's room, with a good monochord and some kind of transfer instrument. But the great book of modes?

Knowing that the tetrachord is the module with which several major civilizations assemble modes, John and I had begun to wonder about how many usable tetrachords there might be. We decided that the ratio 81/80 is the "flip-over" point and the limit of musical use, although not of theoretical use. This is the interval that everyone constantly shifts around when singing or playing major and minor diatonic modes, for it is the difference between a major major second (9/8) and a minor major second (10/9) and the distribution of these two kinds of seconds determines the modal characters. Thus our choice.

John immediately began a program, and began to list results. I think that he used a computer and he soon had quite a list. From his wide reading he also gave attributions as historically documented formations turned up. It was enthralling, and this was indeed the "Great Book"— to my mind the most important work of musical theory since Europe's Renaissance, and probably since the Roman Empire.

But it has taken many years to mature. Not only is John a busy scientist and teacher, but he has wished to bring advanced mathematical thought to the work and enjoys lattice thinking and speculation, often fruitful. He tried a few written introductions which I in turn tried to make intelligible to advanced musicians, who, I thought, might see in his work a marvelous extension of humanist enquiry. Always he found my effort lacking to his needs. He often employed a style of scientese as opaque

to me as his handwriting is illegible. About the latter there is near universal agreement—John himself jestingly joins in this.

In the last very few years all of us have finally had translations into English of Boethius, Ptolemy, and others—all for the first time in our language. For decades before this John worked from the Greek and other languages. This, too, was formidable.

Few studies have stimulated me as has John Chalmers's *Divisions of the Tetrachord*. It is a great work by any standards, and I rejoice.

LOU HARRISON

PREFACE

THIS BOOK IS WRITTEN to assist the discovery of new musical resources, not to reconstruct the lost musical culture of ancient Greece. I began writing it as an annotated catalog of tetrachords while I was a post-doctoral fellow in the Department of Genetics at the University of California, Berkeley in the early 1970s. Much earlier, I had become fascinated with tuning theory while in high school as a consequence of an unintelligible and incorrect explanation of the 12-tone equal temperament in a music appreciation class. My curiosity was aroused and I went to the library to read more about the subject. There I discovered Helmholtz's *On the Sensations of Tone* with A. J. Ellis's annotations and appendices, which included discussions of non-12-tone equal temperaments and long lists of just intervals and historical scales. Later, the same teacher played the 1936 Havana recording of Julián Carrillo's *Preludio a Colón* to our class, ostensibly to demonstrate the sorry condition of modern music, but I found the piece to be one of almost supernatural beauty, and virtually the only interesting music presented the entire semester.

During the next summer vacation, I made a crude monochord calibrated to 19-tone equal temperament, and later some pan pipes in the 5- and 9-tone equal systems. Otherwise, my interest in microtonal music remained more or less dormant for lack of stimulation until as a sophomore at Stanford I attended its overseas campus in Stuttgart. Music appreciation happened to be one of the required courses and Stockhausen was invited to address the class and play tapes of "elektronische Musik," an art-form totally unknown to me at the time. This experience rekindled my interest in

music theory and upon my return to California, I tried to sign up for courses in experimental music. This proved impossible to do, but I did find Harry Partch's book and a recording of the complete *Oedipus* in the Music Library. Thus I began to study microtonal tuning systems. My roommates were astonished when I drove nails into my desk, strung guitar strings between them, and cut up a broom handle for bridges, but they put up with the resulting sounds more or less gracefully.

During my first year of graduate school in biology at UCSD, I came across the article by Tillman Schafer and Jim Piehl on 19-tone instruments (Schafer and Piehl 1947). Through Schafer, who still lived in San Diego at that time, I met Ivor Darreg and Ervin Wilson. Later Harry Partch joined the UCSD music faculty and taught a class which I audited in 1967–68. About this time also, I began collaborating with Ervin Wilson on the generation of equal temperament and just intonation tables at the UCSD computer center (Chalmers 1974, 1982).

After finishing my Ph.D., I received a post-doctoral fellowship from the National Institutes of Health to do research at the University of Washington in Seattle and from there I moved to Berkeley to the Department of Genetics to continue attempting to study cytoplasmic or non-Mendelian genetics in the mold *Neurospora crassa*. A visit by John Grayson provided an opportunity to drive down to Aptos and meet Lou Harrison. I mentioned to Lou that I had begun a list of tetrachords in an old laboratory notebook and he asked me for a copy.

I photocopied the pages for him and mailed them immediately. Lou urged me to expand my notes into a book about tetrachords, but alas, a number of moves and the demands of a career as both an industrial and academic biologist competed with the task. While working for Merck Sharp & Dohme in New Jersey before moving to Houston in the mid-1970s, I wrote a first and rather tentative draft. I also managed to find the time to edit and publish *Xenharmonikon, An Informal Journal of Experimental Music*, while certain harmonic ideas gestated, but I had to suspend publication in 1979. Happily, it was resurrected in 1986 by Daniel Wolf and I resumed the editorship late in 1989.

In the winter of 1980, I was invited to the Villa Serbelloni on Lake Como by the Rockefeller Foundation to work on the book and I completed another draft there. Finally, through the efforts of Larry Polansky and David Rosenboom, I was able to spend the summer of 1986 at Mills College

working on the manuscript.

It was at Mills also that I discovered that the Macintosh computer has four voices with excellent pitch resolution and is easily programmed in BASIC to produce sound. This unexpected opportunity allowed me to generate and hear a large number of the tetrachords and to test some of my theories, resulting in a significant increase in the size of the Catalog and much of the material in chapter 7.

After returning to Houston to work for a while as a consultant for a biotechnology firm, I moved back to Berkeley in the fall of 1987 so that I could devote the necessary time to completing the book. With time out to do some consulting, learn the HMSL music composition and performing language developed at Mills College, and work as a fungal geneticist once again at the University of California, the book was finally completed.

A few words on the organization of this work are appropriate. The first three chapters are concerned with tetrachordal theory from both classical Graeco-Roman and to a lesser extent medieval Islamic perspectives. The former body of theory and speculation have been discussed *in extenso* by numerous authorities since the revival of scholarship in the West, but the latter has not, as yet, received the attention it deserves from experimentally minded music theorists.

After considerable thought, I have decided to retain the Greek nomenclature, though not the Greek notation. Most importantly, it is used in all the primary and secondary sources I have consulted; readers desiring to do further research on tetrachords will have become familiar with the standard vocabulary as a result of exposure to it in this book. Secondly, the Greek names of the modes differ from the ecclesiastical ones used in most counterpoint classes. To avoid confusion, it is helpful to employ a consistent and unambiguous system, which the Greek terminology provides.

Since many of the musical concepts are novel and the English equivalents of a number of the terms have very different meanings in traditional music theory, the Greek terminology is used throughout. For example, in Greek theory, the adjective *enharmonic* refers to a type of tetrachord containing a step the size of a major third, with or without the well-known microtones. In the liturgical music theory of the Greek Orthodox church, also called Byzantine (Savas 1965; Athanasopoulos 1950), it refers to varieties of diatonic and chromatic tunings, while in traditional European theory, it refers to two differently written notes with the same pitch. Where

modern terms are familiar and unambiguous, and for concepts not part of ancient Greek music theory, I have used the appropriate contemporary technical vocabulary.

Finally, I think the Greek names add a certain mystique or glamour to the subject. I find the sense of historical continuity across two and a half millennia exhiliarating—four or more millennia if the Babylonian data on the diatonic scale are correct (Duchesne-Guillemin 1963; Kilmer 1960). Harry Partch must have felt similarly when he began to construct the musical system he called *monophony* (Partch [1949] 1974). Science, including experimental musicology, is a cumulative enterprise; it is essential to know where we have been, as we set out on new paths. Revolutions do not occur *in vacuo*.

The contents of the historical chapters form the background for the new material introduced in chapters 4 through 7. It is in these chapters that nearly all claims for originality and applicability to contemporary composition reside. In particular, chapters 5, 6, and 7 are intended to be of assistance to composers searching for new *materia musica*.

Chapter 8 deals with the heterodox, though fascinating, speculations of Kathleen Schlesinger and some extrapolations from her work. While I do not believe that her theories are descriptive of Greek music at any period, they may serve as the basis for a coherent approach to scale construction independent of their historical validity.

While not intended as a comprehensive treatise on musical scale construction, for which several additional volumes at least as large as this would be required, this work may serve as a layman's guide to the tetrachord and to scales built from tetrachordal modules. With this in mind, a glossary has been provided which consists of technical terms in English pertaining to intonation theory and Greek nomenclature as far as it is relevant to the material and concepts presented in the text. Terms explained in the glossary are italicized at their first appearance in the text.

The catalogs of tetrachords in chapter 9 are both the origin of the book and its justification—the first eight chapters could be considered as an extended commentary on these lists.

ACKNOWLEDGMENTS

PORTIONS OF CHAPTER 5 and an earlier version of chapter 6 originally appeared in the journal *Xenharmonikon* (Chalmers 1975; 1989). A much shorter draft of the book was written at the Centro Culturale Della Fondazione Rockefeller at Bellagio, Italy while I was a Scholar-in-Residence in 1980. I would like to express my gratitude to Larry Polansky and David Rosenboom for arranging a summer residency for me at Mills College in 1986 to work on the manuscript, and for introducing me to the Macintosh as a word processor and acoustic workstation.

Thanks are also due to Dr. Patricia St. Lawrence for the opportunity to come to Berkeley and work at the Department of Genetics during the academic years 1987–88 and 1988–89.

Parts of this book are based on the unpublished work of Ervin M. Wilson who not only placed his notes at my disposal but also served as a teacher and critic in the early stages of the manuscript. Any errors or omissions in the presentation of his material are solely my fault. The same may be said of David Rothenberg, whose perception theories are a prominent part of chapter 5.

Finally, it was Lou Harrison who suggested that I write a book on tetrachords in the first place and who has patiently awaited its completion.

1 The tetrachord in experimental music

WHY, IN THE LAST quarter of the twentieth century, would someone write a lengthy treatise on a musical topic usually considered of interest only to students of classical Greek civilization? Furthermore, why might a reader expect to gain any information of relevance to contemporary musical composition from such a treatise? I hope to show that the subject of this book is of interest to composers of new music.

The familiar tuning system of Western European music has been inherited, with minor modifications, from the Babylonians (Duchesne-Guillemin 1963). The tendency within the context of Western European "art music" to use intervals outside this system has been called *microtonality*, *experimental intonation* (Polansky 1987a), or *xenharmonics* (a term proposed by Ivor Darreg). Interest in and the use of microtonality, defined by scalar and harmonic resources other than the traditional 12-tone equal temperament, has recurred throughout history, notably in the Renaissance (Vicentino 1555) and most recently in the late nineteenth and early twentieth century. The converse of this definition is that music which can be performed in 12-tone equal temperament without significant loss of its identity is not truly *microtonal*. Moreover, the musics of many of the other cultures of the world are microtonal (in relation to 12-tone equal temperament) and European composers have frequently borrowed musical materials from other cultures and historical periods, such as the Ottoman Empire and ancient Greece.

We owe our traditions of musical science to ancient Greece, and the theoretical concepts and materials of ancient Greek music are basic to an

understanding of microtonal music. Greek musical theory used the *tetrachord* as a building block or module from which scales and *systems* could be constructed. A current revival of interest in microtonality, fueled by new musical developments and technological improvements in computers and synthesizers, makes the ancient tetrachord increasingly germane to contemporary composition.

Contemporary microtonality

Although 12-tone equal temperament became the standard tuning of Western music by the mid-nineteenth century (Helmholtz [1877] 1954), alternative tuning systems continued to find partisans. Of these systems, perhaps the most important was that of Bosanquet (Helmholtz [1877] 1954; Bosanquet 1876), who perfected the generalized keyboard upon which the fingering for musical patterns is invariant under transposition. He also championed the 53-tone equal temperament. Of nineteenth-century theorists, Helmholtz and his translator and annotator A. J. Ellis (Helmholtz [1877] 1954) are outstanding for their attempts to revive the use of just intonation.

The early twentieth century saw a renewed interest in *quarter-tones* (24-tone equal temperament) and other equal divisions of the octave. The Mexican composer Julián Carrillo led a crusade for the equal divisions which preserved the *whole tone* (*zero modulo 6* divisions) through 96-tone temperament or sixteenths of tones. Other microtonal, mostly quarter-tone, composers of note were Alois Hába (Czechoslovakia), Ivan Wyschnegradsky (France), and Mildred Couper (USA). The Soviet Union had numerous microtonal composers and theorists, including Georgy Rimsky-Korsakov, Leonid Sabaneev, Arseny Avraamov, E.K. Rosenov, A.S. Obolovets, and P.N. Renchitsky, before Stalin restrained revolutionary creativity under the doctrine of Socialist Realism (Carpenter 1983). Joseph Yasser (USA) urged the adoption of 19-tone equal temperament and Adriaan Fokker (Holland) revived the theories of his countryman, Christian Huygens, and promoted 31-tone equal temperament. More recently, Martin Vogel in Bonn and Franz Richter Herf in Salzburg have been active in various microtonal systems, the latter especially in 72-tone equal temperament.

No discussion of alternative tunings is complete without mentioning Harry Partch, an American original who singlehandedly made extended

just intonation and home-built instruments not only acceptable, but virtually mandatory for musical experimenters at some stage in their careers. Composers influenced by him include Lou Harrison, Ben Johnston, James Tenney, and younger composers such as Larry Polansky, Cris Forster, Dean Drummond, Jonathan Glasier, and the members of the Just Intonation Network.

Ivor Darreg is an American composer working in California. He has been very actively involved with alternative tunings and new instrument design for more than five decades. Darreg has employed both non-12-tone equal temperaments and various forms of just intonation in his music, theoretical writings, and instruments. More recently, he has begun to use MIDI synthesizers and has explored all the equal temperaments up to 53 tones per octave in a series of improvisations in collaboration with Brian McLaren.

Ervin Wilson is one of the most prolific and innovative inventors of new musical materials extant and has been a major influence on me as well as a source for many tetrachords and theoretical ideas. He holds patents on two original generalized keyboard designs. Wilson has collaborated with Kraig Grady and other experimental musicians in the Los Angeles area. He also assisted Harry Partch with the second edition of *Genesis of a Music* by drawing some of the diagrams in the book.

Some other North American microtonal composers are Ezra Sims, Easley Blackwood, Joel Mandelbaum, Brian McLaren, Arturo Salinas, Harold Seletsky, Paul Rapoport, William Schottstaedt, and Douglas Walker.

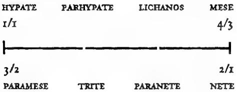
While still very much a minority faction of the contemporary music community, microtonality is rapidly growing. Festivals dedicated to microtonal music have been held in recent years in Salzburg under the direction of Franz Richter Herf; in New York City, produced by Johnny Reinhard; and in San Antonio, Texas, organized by George Cisneros.

Partch, Darreg, Wilson, Harrison, Forster, and William Colvig, among others, have designed and constructed new acoustic instruments for microtonal performance. Tunable electronic synthesizers are now available commercially and provide an an alternative to custom-built acoustic or electroacoustic equipment. A great deal of software, such as *HMSL* from Frog Peak Music, *JICalc* by Robert Rich and Carter Scholz, and Antelope Engineering's *TuneUp*, has been developed to control synthesizers microtonally via MIDI.

Good references for additional information on the history of microtonal systems are Helmholtz ([1877] 1954), Barbour (1951), Partch ([1949] 1974), and Mandelbaum (1961). Small press publications are a rich source and several journals devoted to music in alternative tunings have been published. The major ones are *Xenharmonikon, Interval, Pitch*, and *1/1: The Journal of the Just Intonation Network*. Finally, *Musical Six-Six Bulletin*, *Leonardo: The International Journal of Arts, Science, and Technology*, *Experimental Musical Instruments*, and *Musicworks* have also contained articles about instruments in non-traditional tuning systems.

The tetrachord in microtonal music

Tetrachords are modules from which more complex scalar and harmonic structures may be built. These structures range from the simple heptatonic scales known to the classical civilizations of the eastern Mediterranean to experimental gamuts with many tones. Furthermore, the traditional scales of much of the world's music, including that of Europe, the Near East, the Catholic and Orthodox churches, Iran, and India, are still based on tetrachords. Tetrachords are thus basic to an understanding of much of the world's music.



1-1. *The tetrachord.*

The tetrachord is the interval of a *perfect fourth*, the *diatessaron* of the Greeks, divided into three subintervals by the interposition of two additional notes.

The four notes, or strings, of the tetrachord were named *hypate*, *parhypate*, *lichanos*, and *mese* in ascending order from 1/1 to 4/3 in the first tetrachord of the central octave of the *Greater Perfect System*, the region of the scale of most concern to theorists. Ascending through the second tetrachord, they were called *paramese*, *trite*, *paranete*, and *nete*. (Chapter 6 discusses Greek scales and nomenclature.)

Depending upon the spacing of these interposed tones, three primary genera may be distinguished: the *diatonic*, composed of *tones* and *semitones;* the *chromatic*, of semitones and a *minor third;* and the *enharmonic*, with a *major third* and two quarter-tones. Nuances or *chroai* (often translated "shades") of these primary forms are further characterized by the exact tuning of these intervals.

These four tones apparently sufficed for the recitation of Greek epic poetry, but soon afterwards another tetrachord was added to create a heptachord. As a feeling for the octave developed, the *gamut* was completed,

and from this gamut various sections were later identified and given ancient tribal names (Dorian, Phrygian, et cetera). These *octave species* became the *modes*, two of which, the Lydian and Hypodorian, in the diatonic genus form the basis for the European tonal idiom. Although a formal nomenclature based on the position of the strings later developed, the four tetrachordal tones remained the basis for the Greek solfège: the syllables τε, τω, τη, τα, (pronounced approximately teh, toe, tay, and tah in English) were sung in descending order to the notes of every genus and shade.

The detailed history of the Greek tetrachordal scales is somewhat more complex than the sketchy outline given above. According to literary testimony supported at least in part by archaeology, the diatonic scale and its tuning by a cycle of *perfect fifths*, fourths, and octaves was brought from Egypt (or the Near East) by Pythagoras. In fact the entire *12-tone chromatic* scale in this tuning is thought to have been known to the Babylonians by the second millennium BCE and was apparently derived from earlier Sumerian precursors (Duchesne-Guillemin 1963, 1969; Kilmer 1960). Having arrived in Greece, this scale and its associated tuning doctrines were mingled with local musical traditions, most probably *pentatonic*, to produce a plethora of scale-forms, melody-types and styles (see chapter 6). From a major-third pentatonic, the enharmonic genus can be derived by splitting the semitone (Winnington-Ingram 1928; Sachs 1943). The chromatic genera, whose use in tragedy dates from the late fifth century, may be relicts of various *neutral* and minor-third pentatonics, or conversely, descended from the earlier enharmonic by a process of "sweetening" whereby the pitch of the third tone was raised from a probable 256/243 to produce the more or less consonant intervals 5/4, 6/5, 7/6 and possibly 11/9 (Winnington-Ingram 1928).

The resulting scales were rationalized by the number theory of Pythagoras (Crocker 1963, 1964, 1966) and later by the geometry of Euclid (Crocker 1966; Winnington-Ingram 1932, 1936) to create the body of theory called *harmonics*, which gradually took on existence as an independent intellectual endeavor divorced from musical practice. The acoustic means are now available, and the prevailing artistic ideology is sympathetic enough to end this separation between theory and practice.

Many composers have made direct use of tetrachordal scales in recent compositions. Harry Partch used the pentatonic form of the enharmonic (16/15 · 5/4 · 9/8 · 16/15 · 5/4) in the first of his *Two Studies on Ancient Greek*

Scales (1946) and the microtonal form in the second (in Archytas's tuning, 28/27 · 36/35 · 5/4). Partch also employed this latter scale in *The Dreamer that Remains,* and in verse fifteen of *Petals.* His film score *Windsong* (1958) employs Ptolemy's equable diatonic *(diatonon homalon).* Ivor Darreg's *On the Enharmonic Tetrachord* from his collection *Excursion into the Enharmonic,* was composed in 1965 and published in *Xenharmonikon 3* in 1975. Lou Harrison has used various tetrachords as motives in his "free style" piece *A Phrase for Arion's Leap (Xenharmonikon 3,* 1975). An earlier piece, *Suite* (1949) was based on tetrachords in 12-tone equal temperament. Larry London published his *Eight Pieces for Harp in Ditone Diatonic* in *Xenharmonikon 6* (1977) and his *Four Pieces in Didymus's Chromatic* in *Xenharmonikon 7+8* (1979). In 1984, he wrote a *Suite for Harp* whose four movements used Archytas's enharmonic and a chromatic genus of J.M. Barbour. Gino Robair Forlin's song in Spanish and Zapotec, *Las Tortugas* (1988), is based on the tetrachord 16/15 · 15/14 · 7/6. There are of course many other recent pieces less explicitly tetrachordal whose pitch structures could be analyzed in tetrachordal terms, but doing so would be a major project outside the scope of this book. Similarly, there is a vast amount of music from Islamic cultures, Hindustani, and Eastern Orthodox traditions which is also constructed from tetrachordal scales. These will not be discussed except briefly in terms of their component tetrachords.

A psychological motivation for the consideration of tetrachords is provided by the classic study of George A. Miller, who suggested that musical scales, in common with other perceptual sets, should have five to nine elements for intuitive comprehension (Miller 1956). Scales with cardinalities in this range are easily generated from tetrachords (chapter 6) and the persistence of tetrachordal scales alongside the development of triad-based harmony may reflect this property.

Tetrachords and their scale-like complexes and aggregates have an intellectual fascination all their own, a wealth of structure whose seductive intricacy I hope to convey in this book.

2 Pythagoras, Ptolemy, and the arithmetic tradition

GREEK MUSICAL TRADITION begins in the sixth century BCE with the semi-legendary Pythagoras, who is credited with discovering that the frequency of a vibrating string is inversely proportional to its length. This discovery gave the Greeks a means to describe musical intervals by numbers, and to bring to acoustics the full power of their arithmetical science. While Pythagoras's own writings on music are lost, his tuning doctrines were preserved by later writers such as Plato, in the *Timaeus*, and Ptolemy, in the *Harmonics*. The scale derived from the *Timaeus* is the so-called Pythagorean tuning of Western European theory, but it is most likely of Babylonian origin. Evidence is found not only in cuneiform inscriptions giving the tuning order, but apparently also as music in a diatonic *major mode* (Duchesne-Guillemin 1963, 1969; Kilmer 1960; Kilmer et al. 1976). This scale may be tuned as a series of perfect fifths (or fourths) and octaves, having the ratios 1/1 9/8 81/64 4/3 3/2 27/16 243/128 2/1, though the Babylonians did not express musical intervals numerically.

The next important theorist in the Greek arithmetic tradition is Archytas, a Pythagorean from the Greek colony of Tarentum in Italy. He lived about 390 BCE and was a notable mathematician as well. He explained the use of the arithmetic, geometric, and harmonic means as the basis of musical tuning (Makeig 1980) and he named the *harmonic mean*. In addition to his musical activities, he was renowned for having discovered a three-dimensional construction for the extraction of the cube root of two.

Archytas is the first theorist to give ratios for all three genera. His tunings are noteworthy for employing ratios involving the numbers 5 and 7

instead of being limited to the 2 and 3 of the orthodox Pythagoreans, for using the ratio 28/27 as the first interval (hypate to parhypate) in all three genera, and for employing the consonant major third, 5/4, rather than the harsher *ditone* 81/64, as the upper interval of the enharmonic genus. These tunings are shown in 2-1.

Other characteristics of Archytas's tunings are the smaller second interval of the enharmonic (36/35 is less than 28/27) and the complex second interval of his chromatic genus.

Archytas's enharmonic is the most consonant tuning for the genus, especially when its first interval, 28/27, is combined with a tone 9/8 below the tonic to produce an interval of 7/6. This note, called *hyperhypate*, is found not only in the *harmoniai* of Aristides Quintilianus (chapter 6), but also in the extant musical notation fragment from the first *stasimon* of Euripides's *Orestes*. It also occurs below a chromatic *pyknon* in the second Delphic hymn (Winnington-Ingram 1936). This usage strongly suggests that the second note of the enharmonic and chromatic genera was not a grace note as has been suggested, but an independent degree of the scale (ibid.). Bacchios, a much later writer, calls the interval formed by the skip from hyperhypate to the second degree an *ekbole* (Steinmayer 1985), further affirming the historical correctness of Archytas's tunings.

The complexity of Archytas's chromatic genus demands an explanation, as Ptolemy's soft chromatic *(chroma malakon)* 28/27 · 15/14 · 6/5 would seem to be more consonant. Evidently the chromatic pyknon still spanned the 9/8 at the beginning of the fourth century, and the 32/27 was felt to be

	ARCHYTAS'S GENERA	
28/27 · 36/35 · 5/4	63 + 49 + 386	ENHARMONIC
28/27 · 243/224 · 32/27	63 + 141 + 294	CHROMATIC
28/27 · 8/7 · 9/8	63 + 231 + 204	DIATONIC
	ERATOSTHENES'S GENERA	
40/39 · 39/38 · 19/15	44 + 45 + 409	ENHARMONIC
20/19 · 19/18 · 6/5	89 + 94 + 316	CHROMATIC
256/243 · 9/8 · 9/8	90 + 204 + 204	DIATONIC
	DIDYMOS'S GENERA	
32/31 · 31/30 · 5/4	55 + 57 + 386	ENHARMONIC
16/15 · 25/24 · 6/5	112 + 71 + 316	CHROMATIC
16/15 · 10/9 · 9/8	112 + 182 + 204	DIATONIC

2-1. *Ptolemy's catalog of historical tetrachords, from the* Harmonics *(Wallis 1682). The genus 56/55 · 22/21 · 5/4 (31 + 81 + 386 cents) is also attributed to Ptolemy. Wallis says that this genus is in all of the manuscripts, but is likely to be a later addition. The statements of Avicenna and Bryennios that 46/45 is the smallest melodic interval supports this view.*

the proper tuning for the interval between the upper two tones. This may be in part because 32/27 makes a 4/3 with the *disjunctive tone* immediately following, but also because the melodic contrast between the 32/27 at the top of the tetrachord and the 7/6 with the hyperhypate below is not as great as the contrast between lower 7/6 and the upper 6/5 of Ptolemy's tuning.

Archytas's diatonic is also found among Ptolemy's own tunings (2-2) and appears in the *lyra* and *kithara* scales that Ptolemy claimed were in common practice in Alexandria in the second century CE. According to Winnington-Ingram (1932), it is even grudgingly admitted by Aristoxenos and thus would appear to have been the principal diatonic tuning from the fourth century BCE through the second CE, a period of some six centuries.

Archytas's genera represent a considerable departure from the austerity of the older Pythagorean forms:

ENHARMONIC: 256/243 · 81/64
CHROMATIC: 256/243 · 2187/2048 · 32/27
DIATONIC: 256/243 · 9/8 · 9/8

The enharmonic genus is shown as a *trichord* because the tuning of the enharmonic genus before Archytas is not precisely known. The semitone was initially undivided and may not have had a consistent division until the stylistic changes recorded in his tunings occurred. In other words, the *incomposite ditone*, not the incidental microtones, is the defining characteristic of the enharmonic genus.

The chromatic tuning is actually that of the much later writer Gaudentius (Barbera 1978), but it is the most plausible of the Pythagorean chromatic tunings.

The diatonic genus is the tuning associated with Pythagoras by all the authors from ancient times to the present (Winnington-Ingram 1932).

2-2. *Ptolemy's own tunings.*

46/45 · 24/23 · 5/4	38 + 75 + 386	ENHARMONIC
28/27 · 15/14 · 6/5	63 + 119 + 316	SOFT CHROMATIC
22/21 · 12/11 · 7/6	81 + 151 + 267	INTENSE CHROMATIC
21/20 · 10/9 · 8/7	85 + 182 + 231	SOFT DIATONIC
28/27 · 8/7 · 9/8	63 + 231 + 204	DIATONON TONIAION
256/243 · 9/8 · 9/8	90 + 204 + 204	DIATONON DITONIAION
16/15 · 9/8 · 10/9	112 + 204 + 182	INTENSE DIATONIC
12/11 · 11/10 · 10/9	151 + 165 + 182	EQUABLE DIATONIC

Ptolemy and his predecessors in Alexandria

In addition to preserving Archytas's tunings, Ptolemy (ca. 160 CE) also transmitted the tunings of Eratosthenes and Didymos, two of his predecessors at the library of Alexandria (2-1). Eratosthenes's (third century BCE) enharmonic and chromatic genera appear to have been designed as simplifications of the Pythagorean prototypes. The use of 40/39 and 20/19 for the lowest interval presages the remarkable *Tanbur of Baghdad* of Al-Farabi with its *subharmonic* division by the *modal determinant* 40 (Ellis 1885; D'Erlanger 1935) and some of Kathleen Schlesinger's speculations in *The Greek Aulos* (1939).

Didymos's enharmonic seems to be mere formalism; the enharmonic genus was extinct in music as opposed to theory by his time (first century BCE). His 1:1 *linear division* of the pyknon introduces the prime number 31 into the musical relationships and deletes the prime number 7, a change which is not an improvement harmonically, though it would be of less significance in a primarily melodic music. His chromatic, on the other hand, is the most consonant non-*septimal* tuning and suggests further development of the musical styles which used the chromatic genus. Didymos's diatonic is a permutation of Ptolemy's intense diatonic (diatonon syntonon). It seems to be transitional between the Pythagorean (*3-limit)* and *tertian* tunings.

Ptolemy's own tunings stand in marked contrast to those of his predecessors. In place of the more or less equal divisions of the pyknon in the genera of the earlier theorists, Ptolemy employs a roughly 1:2 melodic proportion. He also makes greater use of *superparticular* or *epimore* ratios than his forerunners; of his list, only the traditional Pythagorean diatonon ditoniaion contains *epimeres*, which are ratios of the form $(n + m) / n$ where $m > 1$.

The emphasis on superparticular ratios was a general characteristic of Greek musical theory (Crocker 1963; 1964). Only epimores were accepted even as successive consonances, and only the first epimores (2/1, 3/2, and 4/3) were permitted as simultaneous combinations.

There is some empirical validity to these doctrines: there is no question that the first epimores are consonant and that this quality extends to the next group, 5/4 and 6/5, else *tertian harmony* would be impossible. Consonance of the septimal epimore 7/6 is a matter of contention. To my ear, it is consonant, as are the epimeres 7/4 and 7/5 and the inversions of the epimores 5/4 and 6/5 (8/5 and 5/3). Moreover, Ptolemy noticed that octave

compounds of consonances (which are not themselves epimores) were aurally consonant. It is clear, therefore, that it is not just the form of the ratio, but at least two factors, the size of the interval and the magnitude of the defining integers, that determines relative consonance. Nevertheless, there does seem to be some special quality of epimore ratios. I recall a visit to Lou Harrison during which he began to tune a harp to the tetrachordal scale 1/1 27/25 6/5 4/3 3/2 81/50 9/5 2/1. He immediately became aware of the non-superparticular ratio 27/25 by perceiving the lack of resonance in the instrument.

A complete list of all possible tetrachordal divisions containing only superparticular ratios has been compiled by I. E. Hofmann (Vogel 1975). Although the majority of these tetrachords had been discovered by earlier theorists, there were some previously unknown divisions containing very small intervals. The complete set is given in 2-3 and individual entries also appear in the Miscellaneous listing of the Catalog.

The equable diatonic has puzzled scholars for years as it appears to be an academic exercise in musical arithmetic. Ptolemy's own remarks rebut this interpretation as he describes the scale as sounding rather strange or foreign and rustic (ξενικοτερον μεν ποσ και αγροικοτερον, Winnington-Ingram 1932). Even a cursory look at ancient and modern Islamic scales from the Near East suggests that, on the contrary, Ptolemy may have heard a similar scale and very cleverly rationalized it according to the tenets of Greek theory. Such scales with 3/4-tone intervals may be related to

2-3. *Hofmann's list of completely superparticular divisions. This table has been recomposed after Hofmann from Vogel (1975). See Main Catalog for further information. (5) has also been attributed to Tartini, but probably should be credited to Pachymeres, a thirteenth-century Byzantine author.*

1.	256/255 · 17/16 · 5/4	NEW ENHARMONIC
2.	136/135 · 18/17 · 5/4	NEW ENHARMONIC
3.	96/95 · 19/18 · 5/4	WILSON'S ENHARMONIC
4.	76/75 · 20/19 · 5/4	AUTHOR'S ENHARMONIC
5.	64/63 · 21/20 · 5/4	SERRE'S ENHARMONIC
6.	56/55 · 22/21 · 5/4	PSEUDO-PTOLEMAIC ENHARMONIC
7.	46/45 · 24/23 · 5/4	PTOLEMY'S ENHARMONIC
8.	40/39 · 26/25 · 5/4	AVICENNA'S ENHARMONIC
9.	28/27 · 36/35 · 5/4	ARCHYTAS'S ENHARMONIC
10.	32/31 · 31/30 · 5/4	DIDYMOS'S ENHARMONIC
11.	100/99 · 11/10 · 6/5	NEW CHROMATIC
12.	55/54 · 12/11 · 6/5	BARBOUR
13.	40/39 · 13/12 · 6/5	BARBOUR
14.	28/27 · 15/14 · 6/5	PTOLEMY'S SOFT CHROMATIC
15.	16/15 · 25/24 · 6/5	DIDYMOS'S CHROMATIC
16.	20/19 · 19/18 · 6/5	ERATOSTHENES'S CHROMATIC
17.	64/63 · 9/8 · 7/6	BARBOUR
18.	36/35 · 10/9 · 7/6	AVICENNA
19.	22/21 · 12/11 · 7/6	PTOLEMY'S INTENSE CHROMATIC
20.	16/15 · 15/14 · 7/6	AL-FARABI
21.	49/48 · 8/7 · 8/7	AL-FARABI
22.	28/27 · 8/7 · 9/8	ARCHYTAS'S DIATONIC
23.	21/20 · 10/9 · 8/7	PTOLEMY'S SOFT DIATONIC
24.	14/13 · 13/12 · 8/7	AVICENNA
25.	16/15 · 19/18 · 10/9	PTOLEMY'S INTENSE DIATONIC
26.	12/11 · 11/10 · 10/9	PTOLEMY'S EQUABLE DIATONIC

2-4. *Genesis of the enharmonic pykna by katapyknosis. In principle, all pyknotic divisions can be generated by this process, although very high multipliers may be necessary in some cases. The ones shown are merely illustrative. See the Catalogs for the complete list. (1x) The basic form is the enharmonic trichord, or major third pentatonic, often ascribed to Olympos. (2x) Didymos's enharmonion, a "weak" form. (3x) Ptolemy's enharmonion, a "strong" form. To comply with Greek melodic canons, it was reordered as 46/45 · 24/23 · 5/4. (4x) Serre's enharmonic, sometimes attributed to Tartini, and discussed by Perrett (1926, 26). Pachymeres may be the earliest source. (5x) Author's enharmonic, also on Hofmann's list of superparticular divisions. (6x) Wilson's enharmonic, also on Hofmann's list of superparticular divisions.*

INDEX	NUMBERS							PYKNA
1x	16						15	16/15
2x	32			31			30	32/31 · 31/30
3x	48		47		46		45	24/23 · 46/45
4x	64	63		62		61	60	64/63 · 21/20
5x	80	79	78	77	76		75	20/19 · 76/75
6x	96	95	94	93	92	91	90	96/95 · 19/18

Aristoxenos's hemiolic chromatic and may descend from neutral third pentatonics such as Winnington-Ingram's reconstruction of the *spondeion* or libation mode (Winnington-Ingram 1928 and chapter 6), if Sachs's ideas on the origin of the genera have any validity (Sachs 1943). In any case, the scale is a beautiful sequence of intervals and has been used successfully by both Harry Partch *(Windsong, Daphne of the Dunes)* and Lou Harrison, the latter in an improvisation in the early 1970s.

Ptolemy returned to the use of the number seven in his chromatic and soft diatonic genera and introduced ratios of eleven in his intense chromatic and equable diatonic. These tetrachords appear to be in agreement with the musical reality of the era, as most of the scales described as contemporary tunings for the lyra and kithara have septimal intervals (6-4).

Ptolemy's intense diatonic is the basis for Western European just intonation. The Lydian or C mode of the scale produced by this genus is the European major scale, but the *minor mode* is generated by the intervallic retrograde of this tetrachord, 10/9 · 9/8 · 16/15. This scale is not identical to the Hypodorian or A mode of 12-tone equally tempered, meantone, and Pythagorean intonations. (For further discussion of this topic, see chapters 6 and 7.)

The numerical technique employed by Eratosthenes, Didymos, and Ptolemy to define the majority of their tetrachords is called *linear division* and may be identified with the process known in Greek as *katapyknosis.* Katapyknosis consists of the division, or rather the filling-in, of a musical interval by multiplying its numerator and denominator by a set of integers of increasing magnitude. The resulting series of integers between the extreme terms generates a new set of intervals of increasingly smaller span as the multiplier grows larger. These intervals form a series of microtones which are then recombined to produce the desired melodic division, usually composed of epimore ratios. The process may be seen in 2-4 where it is applied to the enharmonic *pyknotic* interval 16:15. By extension, the pyknon may also be termed the katapyknosis (Emmanuel 1921). It consists of three notes, the *barypyknon,* or lowest note, the *mesopyknon,* or middle note, and the *oxypyknon,* or highest.

The harmoniai of Kathleen Schlesinger are the result of applying katapyknosis to the entire octave, 2:1, and then to certain of the ensuing intervals. In chapter 4 it is applied to the fourth to generate *indexed genera.*

The divisions of Eratosthenes and Didymos comprise mainly 1:1 divi-

2-5. *Ptolemy's interpretation of Aristoxenos's genera.*

ENHARMONIC	
40/39 · 39/38 · 19/15	44 + 45 + 409
SOFT CHROMATIC	
30/29 · 29/28 · 56/45	59 + 60 + 379
HEMIOLIC CHROMATIC	
80/77 · 77/74 · 37/30	66 + 69 + 363
INTENSE CHROMATIC	
20/19 · 19/18 · 6/5	89 + 94 + 316
SOFT DIATONIC	
20/19 · 38/35 · 7/6	89 + 142 + 267
INTENSE DIATONIC	
20/19 · 19/17 · 17/15	89 + 192 + 217

sions of the pyknon while those of Ptolemy favor the 1:2 proportion, although in some instances the sub-intervals must be reordered so that the melodic proportions are the canonical order; small, medium and large. This principle was also enunciated by Aristoxenos, but violated by Archytas, Didymos, and Ptolemy himself in his diatonic tunings.

A more direct method of calculating the divisions is to use the following formulae (Winnington-Ingram 1932; Barbera 1978) where x/y is the interval to be linearly divided:

1/1 $2x/(x+y) \cdot (x+y)/2y = x/y$,

1/2 $3x/(2x+y) \cdot (2x+y)/3y = x/y$,

2/1 $3x/(x+2y) \cdot (x+2y)/3y = x/y$.

Finer divisions may be defined analogously; if a/b is the desired proportion and x/y the interval, then $(a+b)\cdot x/(bx+ay)\cdot(bx+ay)/(a+b)\cdot y = x/y$.

The final set of tetrachords given by Ptolemy are his interpretations of the genera of Aristoxenos (2-5). Unfortunately, he seems to have completely misunderstood Aristoxenos's geometric approach and translated his "parts" into aliquot parts of a string of 120 units. Two of the resulting tetrachords are identical to Eratosthenes's enharmonic and chromatic genera, but the others are rather far from Aristoxenos's intent. The Ptolemaic version of the hemiolic chromatic is actually a good approximation to Aristoxenos's soft chromatic. Aristoxenos's theories will be discussed in detail in chapter 3.

The late Roman writers

After Ptolemy's recension of classical tuning lore, a few minor writers such as Gaudentius (fourth century CE) continued to provide tuning information in numbers rather than the fractional tones of the Aristoxenian school. Gaudentius's diatonic has the familiar ditone or Pythagorean tuning, as does his intense chromatic (chroma syntonon), 256/243 · 2187/2048 · 32/27 (Barbera 1978).

The last classical scholar in the ancient arithmetic tradition was the philosopher Boethius (sixth century CE) who added some novel tetrachords and also hopelessly muddled the nomenclature of the modes for succeeding generations of Europeans. Boethius's tuning for the tetrachords in the three principal genera are below:

ENHARMONIC: 512/499 · 499/486 · 81/64

CHROMATIC: 256/243 · 81/76 · 19/16

DIATONIC: 256/243 · 9/8 · 9/8

These unusual tunings are best thought of as a simplification of the Pythagorean forms, as the limma (256/243) is the enharmonic pyknon and the lowest interval of both the chromatic and diatonic genera. The enharmonic uses the 1:1 division formula to divide the 256/243, and the 19/16 is virtually the same size as the Pythagorean minor third, 32/27.

The medieval Islamic theorists

With the exception of Byzantine writers such as Pachymeres, who for the most part repeated classical doctrines, the next group of creative authors are the medieval Islamic writers, Al-Farabi (950 CE), Ibn Sina or Avicenna (1037 CE) and Ṣafiyu-d-Din (1276 CE). These theorists attempted to rationalize the very diverse musics of the Islamic cultural area within the Greek theoretical framework.

In addition to an extended Pythagorean cycle of seventeen tones, genera of divided fifths and a forty-fold division of the the string (Tanbur of Baghdad) in Al-Farabi, several new theoretical techniques are found. Al-Farabi analogizes from the 256/243 · 9/8 · 9/8 of the Pythagorean tuning and proposes reduplicated genera such as 49/48 · 8/7 · 8/7 and 27/25 · 10/9 · 10/9. Avicenna lists other *reduplicated* tetrachords with intervals of approximately 3/4 of a tone and smaller (see the Catalog for these genera). The resemblance of these to Ptolemy's equable diatonic seems more than fortuitous and further supports the notion that *three-quarter-tone* intervals were in actual use in Near Eastern music by Roman times (second century CE). These tetrachords may also bear a genetic relationship to neutral-third pentatonics and to Aristoxenos's hemiolic chromatic and soft diatonic genera as well as Ptolemy's intense chromatic.

Surprisingly, I have been unable to trace the apparently missing reduplicated genus, 11/10 · 11/10 · 400/363 (165 + 165 + 168 cents) that is a virtually equally-tempered division of the 4/3. Lou Harrison has pointed out that tetrachords such as this and the equable diatonic yield scales which approximate the 7-tone equal temperament, an idealization of tuning systems which are widely distributed in sub-Saharan Africa and Southeast Asia.

Other theoretical advances of the Islamic theorists include the use of various arrangements of the intervals of the tetrachords. Safiyu-d-Din listed all six permutations of the tetrachords in his compendious tables, although his work was probably based on Aristoxenos's discussion of the permutations of the tetrachords that occur in the different octave species.

At least for expository purposes, the Islamic theorists favored arrangements with the pyknon uppermost and with the whole tone, when present, at the bottom. This format may be related to the technique of measurement termed *messel*, from the Arabic *al-mithal*, in which the shorter of two string lengths is taken as the unit, yielding numbers in the reverse order of the Greek theorists (Apel 1955, 441–442.).

The so-called *neo-chromatic* tetrachord (Gevaert 1875) with the augmented second in the central position is quite prominent and is also found in some of the later Greek musical fragments and in Byzantine chant (Winnington-Ingram 1936) as the *palace mode*. It is found in the *Hungarian minor* and *Gypsy* scales, but, alas, it has become a common musical cliché, the "snake-charmer's scale" of the background music for exotic Oriental settings on television and in the movies.

The present

After the medieval Islamic writers, there are relatively few theorists expressing any great interest in tetrachords until the nineteenth and twentieth centuries. Notable among the persons attracted to this branch of music theory were Helmholtz ([1877] 1954) and Vogel (1963, 1967, 1975) in Germany; A. J. Ellis (1885), Wilfrid Perrett (1926, 1928, 1931, 1934), R. P. Winnington-Ingram (1928, 1932) and Kathleen Schlesinger (1933) in Britain; Thorvald Kornerup (1934) in Denmark; and Harry Partch (1949) and Ervin Wilson in the United States. The contributions of these scholars and discoverers are listed in the Catalog along with those of many other workers in the arithmetic tradition.

After two and a half millennia, the fascination of the tetrachord has still not vanished. Chapter 4 will deal with the extension of arithmetical techniques to the problem of creating or discovering new tetrachordal genera.

3 Aristoxenos and the geometrization of musical space

ARISTOXENOS WAS FROM the Greek colony of Tarentum in Italy, the home of the famous musician and mathematician Archytas. In the early part of his life, he was associated with the Pythagoreans, but in his later years he moved to Athens where he studied under Aristotle and absorbed the new logic and geometry then being developed (Barbera 1980; Crocker 1966; Litchfield 1988). He was the son of the noted musician Spintharos, who taught him the conservative musical tradition still practiced in the Greek colonies, if not in Athens itself (Barbera 1978).

The geometry of music

The new musical theory that Aristoxenos created about 320 BCE differed radically from that of the Pythagorean arithmeticians. Instead of measuring intervals with discrete ratios, Aristoxenos used continuously variable quantities. Musical notes had ranges and tolerances and were modeled as loci in a continuous linear space. Rather than ascribing the consonance of the octave, fifth, and fourth to the superparticular nature of their ratios, he took their magnitude and consonance as given. Since these intervals could be slightly mistuned and still perceived as categorically invariant, he decided that even the principal consonances of the scale had a narrow, but still acceptable range of variation. Thus, the ancient and bitter controversy over the allegedly unscientific and erroneous nature of his demonstration that the perfect fourth consists of two and one half tones is really inconsequential.

Aristoxenos defined the whole tone as the difference between the two fundamental intervals of the fourth and the fifth, the only consonances smaller than the octave. The octave was found to consist of a fourth and a

fifth, two fourths plus a tone, or six tones. The intervals smaller than the fourth could have any magnitude in principle since they were dissonances and not precisely definable by the unaided ear, but certain sizes were traditional and distinguished the genera known to every musician. These conventional intervals could be measured in terms of fractional tones by the ear alone because musical function, not numerical precision, was the criterion. The tetrachords that Aristoxenos claimed were well-known are shown in 3-1.

Aristoxenos described his genera in units of twelfths of a tone (Macran 1902), but later theorists, notably Cleonides, translated these units into a cipher consisting of 30 parts (*moria*) to the fourth (Barbera 1978). The enharmonic genus consisted of a pyknon divided into two 3-part microtones or *dieses* and a ditone of 24 parts to complete the perfect fourth. Next come three shades of the chromatic with dieses of 4, 4.5, and 6 parts and upper intervals of 22, 21, and 18 parts respectively. The set was finished with two diatonic tunings, a soft diatonic (6 + 9 + 15 parts), and the intense diatonic (6 + 12 + 12 parts). The former resembles a chromatic genus, but the latter is similar to our modern conception of the diatonic and probably

3-1. *The genera of Aristoxenos. The descriptions of Aristoxenos (Macran 1902) in terms of twelfths of tones have been converted to cents, assuming 500 cents to the equally tempered fourth. The interpretation of Aristoxenos's fractional tones as thirty parts to the fourth is after the second century theorist Cleonides.*

ENHARMONIC

0 50 100 500

3 + 3 + 24 PARTS

1/4 + 1/4 + 2 TONES

50 + 50 + 400 CENTS

INTENSE CHROMATIC

0 100 200 500

6 + 6 + 18 PARTS

1/2 + 1/2 + 1 1/2 TONES

100 + 100 + 300 CENTS

SOFT CHROMATIC

0 67 133 500

4 + 4 + 22 PARTS

1/3 + 1/3 + 1 5/6 TONES

67 + 67 + 333 CENTS

SOFT DIATONIC

0 100 250 500

6 + 9 + 15 PARTS

1/2 + 3/4 + 1 1/4 TONES

100 + 150 + 250 CENTS

HEMIOLIC CHROMATIC

0 75 150 500

4.5 + 4.5 + 21 PARTS

3/8 + 3/8 + 1 3/4 TONES

75 + 75 + 350 CENTS

INTENSE DIATONIC

0 100 300 500

6 + 12 + 12 PARTS

1/2 + 1 + 1 TONES

100 + 200 + 200 CENTS

3-2. *Other genera mentioned by Aristoxenos.*

UNNAMED CHROMATIC

0 67 200 500

4 + 8 + 18 PARTS

1/3 + 2/3 + 1 1/2 TONES

67 + 133 + 300 CENTS

DIATONIC WITH SOFT CHROMATIC DIESIS

0 67 300 500

4 + 14 + 12 PARTS

1/3 + 1 1/6 + 1 TONES

67 + 233 + 200 CENTS

DIATONIC WITH HEMIOLIC CHROMATIC DIESIS

0 75 300 500

4.5 + 13.5 + 21 PARTS

3/8 + 1 1/8 + 1 TONES

75 + 225 + 200 CENTS

REJECTED CHROMATIC

0 100 150 500

6 + 3 + 21 PARTS

1/2 + 1/4 + 1 3/4 TONES

100 + 50 + 350 CENTS

UNMELODIC CHROMATIC

0 75 133 500

4.5 + 3.5 + 22 PARTS

3/8 + 7/24 + 1 5/6 TONES

75 + 58 + 367 CENTS

represents the Pythagorean form. Two such 30-part tetrachords and a whole tone of twelve parts completed an octave of 72 parts.

Several properties of the Aristoxenian tetrachords are immediately apparent. The enharmonic and three chromatic genera have small intervals with similar sizes, as if the boundary between the enharmonic and chromatic genus was not yet fixed. The two chromatics between the syntonic chromatic and the enharmonic may represent developments of neutral-third pentatonics mentioned in chapter 2.

The pyknon is always divided equally except in the two diatonic genera whose first intervals (half tones) are the same as that of the syntonic chromatic. Thus Aristoxenos is saying that the first interval must be less than or equal to the second, in agreement with Ptolemy's views nearly five hundred years later.

The tetrachords of 3-2 are even more interesting. The first, an approved but unnamed chromatic genus, not only has the 1:2 division of the pyknon, but more importantly, is extremely close to Archytas's chromatic tuning (Winnington-Ingram 1932). The diatonic with soft chromatic diesis is a very good approximation to Archytas's diatonic as well (ibid.). Only Archytas's enharmonic is missing, though Aristoxenos seems to allude to it in his polemics against raising the second string and thus narrowing the largest interval (ibid.). These facts clearly show that Aristoxenos understood the music of his time.

The last two tetrachords in 3-2 were considered unmusical because the second interval is larger than the first. Winnington-Ingram (1932) has suggested that Aristoxenos could have denoted Archytas's enharmonic tuning as 4 + 3 + 23 parts (67 + 50 + 383), a tuning which suffers from the same defect as the two rejected ones. A general prejudice against intervals containing an odd number of parts may have caused Aristoxenos to disallow tetrachords such as 5 + 11 + 14, 5 + 9 + 16 (ibid.), and 5 + 6 + 19 (Macran 1902).

The alleged discovery of equal temperament

Because a literal interpretation of Aristoxenos's parts implies equal temperaments of either 72 or 144 tones per octave to accommodate the hemiolic chromatic and related genera, many writers have credited him with the discovery of the traditional western European 12-tone intonation. This conclusion would appear to be an exaggeration, at the least. There is

no evidence whatsoever in any of Aristoxenos's surviving writings or from any of the later authors in his tradition that equal temperament was intended (Litchfield 1988).

Greek mathematicians would have had no difficulty computing the string lengths for tempered scales, especially since only two computations for each tetrachord would be necessary, and only a few more for the complete octave scale. Methods for the extraction of the square and cube roots of two were long known, and Archytas, the subject of a biography by Aristoxenos, was renowned for having discovered a three-dimensional construction for the cube root of two, a necessary step for dividing the octave into the 12, 24, 36, 72, or 144 geometric means as required by Aristoxenos's tetrachords (Heath [1921] 1981, 1:246-249). Although irrationals were a source of great worry to Pythagorean mathematicians, by Ptolemy's time various mechanical instruments such as the *mesolabium* had been invented for extracting roots and constructing geometric means (ibid., 2:104). Yet neither Ptolemy nor any other writer mentions equal temperament.

Ptolemy, in fact, utterly missed Aristoxenos's point and misinterpreted these abstract, logarithmic parts as aliquot segments of a real string of 120 units with 60 units at the octave, 80 at the fifth, and 90 at the fourth. His upper tetrachord had only twenty parts, necessitating the use of complicated fractional string lengths to express the actually simple relations in the upper tetrachords of the octave scales.

There are two obvious explanations for this situation. First, Aristoxenos was opposed to numeration, holding that the trained ear of the musician was sufficiently accurate. Second, Greek music was mostly monophonic, with heterophonic rather than harmonic textures. Although modulations and chromaticism did exist, they would not have demanded the paratactical pitches of a tempered gamut (Polansky 1987a). There was no pressing need for equal temperament, and if it was discovered, the fact was not recorded (for a contrary view, see McClain 1978).

Later writers and Greek notation

Although most of the later theorists continued the geometric approach taken by Aristoxenos, they added little to our knowledge of Greek music theory with few exceptions. Cleonides introduced the cipher of thirty parts to the fourth. Bacchios gave the names of some intervals of three and five dieses which were alleged to be features of the ancient style, and Aristides

Quintilianus offered a purported list of the ancient harmoniai mentioned by Plato in the *Timaeus*.

One exception was Alypius, a late author who provided invaluable information on Greek musical notation. His tables of keys or *tonoi* were deciphered independently in the middle of the nineteenth century by Bellermann (1847) and Fortlage (1847), and made it possible for the few extant fragments of Greek music to be transcribed into modern notation and understood. Unfortunately, Greek notation lacked both the numerical precision of the tuning theories, and the clarity of the system of genera and modes (chapter 6). Additionally, there are unresolved questions concerning the choice of alternative, but theoretically equivalent, spellings of certain passages. Contemplation of these problems led Kathleen Schlesinger to the heterodox theories propounded in *The Greek Aulos*.

Others have simply noted that the notation and its nomenclature seem to have evolved away from the music they served until it became an academic subject far removed from musical needs (Henderson 1957). For these reasons, little will be said about notation; knowledge of it is not necessary to understand Greek music theory nor to apply Greek theory to present-day composition.

3-3. *Two medieval Islamic forms. These two medieval Islamic tetrachords are Aristoxenian approximations to Ptolemy's equable diatonic. The Arabs also listed Aristoxenos's other tetrachords in their treatises.*

NEUTRAL DIATONIC

0 200 350 500

12 + 9 + 9 PARTS

1 + 3/4 + 3/4 TONES

200 + 150 + 150 CENTS

EQUAL DIATONIC

0 167 334 500

10 + 10 + 10 PARTS

5/6 + 5/6 + 5/6 TONES

167 + 167 + 166 CENTS

Medieval Islamic theorists

As the Roman empire decayed, the locus of musical science moved from Alexandria to Byzantium and to the new civilization of Islam. Aristoxenos's geometric tradition was appropriated by both the Greek Orthodox church to describe its liturgical modes. Aristoxenian doctrines were also included in the Islamic treatises, although arithmetic techniques were generally employed.

The tetrachords of 3-3 were used by Al-Farabi to express 3/4-tone scales similar to Ptolemy's equable diatonic in Aristoxenian terms. If one subtracts 10 + 10 + 10 parts from Ptolemy's string of 120 units, one obtains the series 120 110 100 90, which are precisely the string lengths for the equable diatonic (12/11 · 11/10 · 10/9). It would appear that the nearly equal tetrachord 11/10 · 11/10 · 400/363 was not intended.

The tetrachord 12 + 9 + 9 yields the permutation 120 108 99 90, or 10/9 · 12/11 · 11/10. This latter tuning is similar to others of Al-Farabi and Avicenna consisting of a tone followed by two 3/4-tone intervals. Other tetrachords of this type are listed in the Catalog.

Eastern Orthodox liturgical music

The intonation of the liturgical music of the Byzantine and Slavonic Orthodox churches is a complex problem and different contemporary authorities offer quite different tunings for the various scales and modes (*echoi*). One of the complications is that until recently a system of 28 parts to the fourth, implying a 68-note octave (28 + 12 + 28 = 68 parts), was in use along with the Aristoxenian 30 + 12 + 30 parts (Tiby 1938).

Another problem is that the nomenclature underwent a change; the term enharmonic was applied to both a neo-chromatic and a diatonic genus, and chromatic was associated with the neo-chromatic forms. Finally, many of the modes are composed of two types of tetrachord, and both chromaticism and modulation are commonly employed in melodies.

Given these complexities, only the component tetrachords extracted from the scales are listed in 3-4. The format of this table differs from that of 3-1 through 3-3 in that the diagrams have been omitted and partially replaced by the ratios of plausible arithmetic forms. The four tetrachords from Tiby which utilize a system of 28 parts to the fourth are removed to the Tempered section of the Catalog.

3-4. *Byzantine and Greek Orthodox tetrachords. Athanasopoulos's enharmonic and diatonic genera consist of various permutations of 6 + 12 + 12, i.e. 12 + 6 + 12. Xenakis permits permutations of the 12 + 11 + 7 and 6 + 12 + 12 genera. A closer, but non-superparticular, approximation to Xenakis's intense chromatic would be 22/21 · 6/5 · 35/33.*

PARTS	CENTS	RATIOS	GENUS
		ATHANASOPOULOS (1950)	
9 + 15 + 6	150 + 250 + 100	—	CHROMATIC
6 + 18 + 6	100 + 300 + 100	—	CHROMATIC
6 + 12 + 12	100 + 200 + 200	—	DIATONIC
12 + 12 + 6	200 + 200 + 100	—	ENHARMONIC
		SAVAS (1965)	
8 + 14 + 8	133 + 233 + 133	—	CHROMATIC
10 + 8 + 12	167 + 133 + 200	—	DIATONIC
8 + 12 + 10	133 + 200 + 167	—	BARYS DIATONIC
12 + 12 + 6	200 + 200 + 100	—	ENHARMONIC
8 + 16 + 6	133 + 267 + 100	—	BARYS ENHARMONIC
6 + 20 + 4	100 + 333 + 67	—	PALACE MODE (NENANO)
		XENAKIS (1971)	
7 + 16 + 7	117 + 266 + 117	16/15 · 7/6 · 15/14	SOFT CHROMATIC
5 + 19 + 6	83 + 317 + 100	256/243 · 6/5 · 135/128	INTENSE CHROMATIC
12 + 11 + 7	200 + 183 + 117	9/8 · 10/9 · 16/15	DIATONIC
6 + 12 + 12	100 + 200 + 200	256/243 · 9/8 · 9/8	ENHARMONIC

The tetrachords of Athanasopoulos (1950) are clearly Aristoxenian in origin and inspiration, despite being reordered. One of his chromatics is Aristoxenos's soft diatonic and the other is Aristoxenos's intense chromatic. The rest of his tetrachords are permutations of Aristoxenos's intense diatonic.

Savas's genera (Savas 1965) may reflect an Arabic or Persian influence, as diatonics with intervals between 133 and 167 cents are reminiscent of Al-Farabi's and Avicenna's tunings (chapter 2 and the Catalog). They may plausibly represent 12/11 and 11/10 so that his diatonic tunings are intended to approximate a reordered Ptolemy's equable diatonic. His chromatic resembles 14/13 · 8/7 · 13/12 and his Barys enharmonic, 15/14 · 7/6 · 16/15. Savas's ordinary enharmonic may stand for either Ptolemy's intense diatonic (10/9 · 9/8 · 16/15) or the Pythagorean version (256/243 · 9/8 · 9/8). The palace mode could be 15/14 · 6/5 · 28/27 (Ptolemy's intense chromatic). The above discussion assumes that some form of just intonation is intended.

The tunings of the experimental composer Iannis Xenakis (1971) are clearly designed to show the continuity of the Greek Orthodox liturgical tradition with that of Ptolemy and the other ancient arithmeticians, though they are expressed in Aristoxenian terms. This continuity is debatable; internal evidence suggests that the plainchant of the Roman Catholic church is derived from Jewish cantillation rather than Graeco-Roman secular music (Idelsohn 1921). It is hard to see how the music of the Eastern church could have had an entirely different origin, given its location and common early history. A case for evolution from a common substratum of Near Eastern music informed by classical Greek theory and influenced by the Hellenized Persians and Arabs could be made and this might give the appearance of direct descent.

The robustness of the geometric approach of Aristoxenos is still evident today after 2300 years. The musicologist James Murray Barbour, a strong advocate of equal temperament, proposed 2 + 14 + 14 and 8 + 8 + 14 as Aristoxenian representations of 49/48 · 8/7 · 8/7 and 14/13 · 13/12 · 8/7 in his 1953 book on the history of musical scales, *Tuning and Temperament*. With Xenakis's endorsement, Aristoxenian principles have become part of the world of international, or transnational, contemporary experimental music. In the next chapter the power of the Aristoxenian approach to generate new musical materials will be demonstrated.

4 The construction of new genera

THIS CHAPTER IS concerned with the construction of new genera in addition to those collated from the texts of the numerous classical, medieval, and recent writers. The new tetrachords are a very heterogeneous group, since they were generated by the author over a period of years using a number of different processes as new methods were learned or discovered. Including historical tetrachords, the tabulated genera in the catalogs number 723, of which 476 belong in the Main Catalog, 16 in the reduplicated section, 101 under miscellaneous, 98 in the tempered list, and 32 in the semi-tempered category.

The genera in the Main Catalog are classified according to the size of their largest or *characteristic interval* (CI) in decreasing order from 13/10 (454 cents) to 10/9 (182 cents). There are 73 CIs acquired from diverse historical and theoretical sources (4-1). Sources are documented in the catalogs. The theoretical procedures for obtaining the new genera are described in this chapter and the next.

New genera derived by linear division

The first of the new genera are those whose CIs are relatively simple non-superparticular ratios such as 11/9, 14/11, and 16/13. These ratios were drawn initially from sources such as Harry Partch's 43-tone, 11-limit just intonation gamut, but it was discovered later that some of these CIs are to be found in historical sources as well. The second group is composed of intervals such as 37/30, which were used sporadically by historical writers. To these ratios may be added their 4/3's and 3/2's complements, e.g. 27/22

4-1. Characteristic intervals (CIs) of new genera in just intonation. The CI is the largest interval of the tetrachord and the pyknon or apyknon is the difference between the CI and the fourth. Because many of the new genera have historically known CIs, all of the CIs in the Main Catalog are listed in this table. The CIs of the reduplicated, miscellaneous, tempered, and semi-tempered lists are not included in this table.

HYPERENHARMONIC GENERA

The term hyperenharmonic is originally from Wilson and refers to genera whose CI is greater than 425 cents. The prototypical hyperenharmonic genus is Wilson's 56/55 · 55/54 · 9/7. See chapter 5 for classification schemes.

	CI	PYKNON	CENTS
H1	13/10	40/39	454 + 44
H2	35/27	36/35	449 + 49
H3	22/17	34/33	446 + 52
H4	128/99	33/32	445 + 53
H5	31/24	32/31	443 + 55
H6	40/31	31/30	441 + 57
H7	58/45	30/29	439 + 59
H8	9/7	28/27	435 + 63
H9	104/81	27/26	433 + 65
H10	50/39	26/25	430 + 68
H11	32/25	25/24	427 + 71

ENHARMONIC GENERA

The CIs of the enharmonic genera range from 375 to 425 cents.

E1	23/18	24/23	424 + 73
E2	88/69	23/22	421 + 77
E3	50/41	160/153	421 + 77
E4	14/11	22/21	418 + 81
E5	80/63	21/20	414 + 84
E6	33/26	104/99	413 + 85
E7	19/15	20/19	409 + 89
E8	81/64	256/243	408 + 90
E9	24/19	19/18	404 + 94
E10	34/27	18/17	399 + 99
E11	113/90	120/113	394 + 104
E12	64/51	17/16	393 + 105
E13	5/4	16/15	386 + 112
E14	8192/6561	2187/2048	384 + 114
E15	56/45	15/14	379 + 119
E16	41/33	44/41	376 + 122

CHROMATIC GENERA

The CIs of the chromatic genera range from 375 to 250 cents.

C1	36/29	29/27	374 + 124
C2	26/21	14/13	370 + 128
C3	21/17	68/63	366 + 132
C4	100/81	27/25	365 + 133
C5	37/30	40/37	363 + 135
C6	16/13	13/12	359 + 139
C7	27/22	88/81	355 + 143
C8	11/9	12/11	347 + 151
C9	39/32	128/117	342 + 156
C10	28/23	23/21	341 + 157
C11	17/14	56/51	336 + 162
C12	40/33	11/10	333 + 165
C13	29/24	32/29	328 + 170
C14	6/5	10/9	316 + 182
C15	25/21	28/25	302 + 196
C16	19/16	64/57	298 + 201
C17	32/27	9/8	294 + 204
C18	45/38	152/135	293 + 205
C19	13/11	44/39	289 + 209
C20	33/28	112/99	284 + 214
C21	20/17	17/15	281 + 217
C22	27/23	92/81	278 + 220
C23	75/64	256/225	275 + 223
C24	7/6	8/7	267 + 231
C25	136/117	39/34	261 + 238
C26	36/31	31/27	259 + 239
C27	80/69	23/20	256 + 242
C28	22/19	38/33	254 + 244
C29	52/45	15/13	250 + 248

DIATONIC GENERA

The CIs of the diatonic genera range from 250 to 166 cents. In the diatonic genera, a pyknon does not exist.

D1	15/13	52/45	248 + 250
D2	38/23	22/19	242 + 256
D3	23/20	80/69	242 + 256
D4	31/27	36/31	239 + 259
D5	39/34	136/117	238 + 261
D6	8/7	7/6	231 + 267
D7	256/225	75/64	223 + 275
D8	25/22	88/75	221 + 277
D9	92/81	27/23	220 + 278
D10	76/67	67/57	218 + 280
D11	17/15	20/17	217 + 281
D12	112/99	33/28	214 + 284
D13	44/39	13/11	209 + 289
D14	152/135	45/38	205 + 293
D15	9/8	32/27	204 + 294
D16	160/143	143/120	194 + 304
D17	10/9	6/5	182 + 316

is the 3/2's complement of 11/9 and 52/45 the 4/3's complement of 15/13. Various genera were then constructed by dividing the pykna or apykna by linear division into two or three parts to produce 1:1, 1:2, and 2:1 divisions. Both the 1:2 and 2:1 divisions were made to locate genera composed mainly of superparticular ratios. Even Ptolemy occasionally had to reorder the intervals resulting from triple division before recombining two of them to produce the two intervals of the pyknon (2-2 and 2-4). More complex divisions were found either by inspection or by katapyknosis with larger multipliers.

4-2. *Indexed genera. The terms 4 and 3 which represent the 1/1 and 4/3 of the final tetrachord are multiplied by the index. The lefthand sets of tetrachords are those generated by selecting and recombining the successive intervals resulting from the additional terms after the multiplication. The righthand sets of tetrachords have been reduced to lowest terms and ordered with the CI uppermost.*

MULTIPLIER: 4	TERMS: **16 15 14 13 12**
16/15 · 15/14 · 14/12	16/15 · 15/14 · 7/6
16/15 · 15/13 · 13/12	16/15 · 13/12 · 15/13
16/14 · 14/13 · 13/12	14/13 · 13/12 · 8/7
MULTIPLIER: 5	TERMS: **20 19 18 17 16 15**
20/19 · 19/18 · 18/15	20/19 · 19/18 · 6/5
20/19 · 19/17 · 17/15	20/19 · 19/17 · 17/15
20/19 · 19/16 · 16/15	20/19 · 16/15 · 19/16
20/18 · 18/17 · 17/15	18/17 · 10/9 · 17/15
20/18 · 18/16 · 16/15	16/15 · 10/9 · 9/8
20/17 · 17/16 · 16/15	17/16 · 16/15 · 20/17
MULTIPLIER: 6	TERMS: **24 23 22 21 20 19 18**
24/23 · 23/22 · 22/18	24/23 · 23/22 · 11/9
24/23 · 23/21 · 21/18	24/23 · 23/21 · 7/6
24/23 · 23/20 · 20/18	24/23 · 10/9 · 23/20
24/23 · 23/19 · 19/18*	24/23 · 19/18 · 23/19
24/22 · 22/21 · 21/18	22/21 · 12/11 · 7/6
24/22 · 22/20 · 20/18	12/11 · 11/10 · 10/9
24/22 · 22/19 · 19/18	19/18 · 12/11 · 22/19
24/21 · 21/20 · 20/18	21/20 · 10/9 · 8/7
24/21 · 21/19 · 19/18	19/18 · 21/19 · 8/7
24/20 · 20/19 · 19/18	20/19 · 19/18 · 6/5

* *see Catalog number 536.*

Indexed genera

One useful technique, originated by Ervin Wilson, is a variation of the katapyknotic process. In 4-2 this technique is applied to the 4/3 rather than to the pyknon (as it was in 2-4). The 1/1 and 4/3 of the undivided tetrachord are expressed as 3 and 4, and are multiplied by a succession of numbers of increasing magnitude. The new terms resulting from such a multiplication and all the intermediate numbers define a set of successive intervals which may be sequentially recombined to yield the three intervals of tetrachords. I have termed the multiplier, the index, and the resulting genera *indexed genera*. The intermediate terms are a sequence of arithmetic means between the extremes.

The major shortcoming of this procedure is that the number of genera grows rapidly with the index. There are 120 genera of index 17, and not all of these are worth cataloguing, since other genera of similar melodic contours and simpler ratios are already known and tabulated. The technique is still of interest, however, to generate sets of tetrachords with common numerical relations for algorithmic composition.

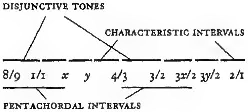
Pentachordal families

Archytas's genera were devised so that they made the interval 7/6 between their common first interval, 28/27, and the note a 9/8 below the first note of the tetrachord (Erickson 1965; Winnington-Ingram 1932; see also 6-1). Other first intervals *(x)* may be chosen so that in combination with the 9/8 they generate harmonically and melodically interesting intervals. These intervals may be termed *pentachordal intervals (PI)* as they are part of a pentachordal, rather than a tetrachordal tonal sequence. Three such groups or families of tetrachords are given in 4-3 along with their initial and pentachordal intervals.

The 28/27 family is an expansion of Archytas's set of genera. The 40/39 family fits quite well into 24-tone equal temperament because of the reasonably close approximation of many of the ratios of 13 to quarter-tone intervals. The 15/13 is another plausible tuning for the interval of five dieses which was reputed to be a feature of the oldest scales (chapter 6; Bacchios, 320 CE in Steinmayer 1985). The 16/15 family contains the most consonant tunings of the chromatic and diatonic genera.

The pentachordal intervals of 4-3 are the *mediants* ("thirds") of the triads which generate the *tritriadic* scales of chapter 7, where they are discussed in greater detail. In general, all tetrachords containing a medial 9/8 may function as generators of tritriadic scales.

4-3. *Pentachordal intervals and families. These tetrachords are defined by two parameters: the pentachordal interval, 9x/8, and the characteristic interval, which determines the genus. An initial interval* x *results in a pentachordal interval (PI) of 9x/8. These pentachordal families are the most important tritriadic genera of chapter 7. The initials are the first intervals of the tetrachords.*



INITIAL	PI	INITIAL	PI	INITIAL	PI
16/15	6/5	10/9	5/4	8/7	9/7
28/27	7/6	12/11	27/22	88/81	11/9
13/12	39/32	128/117	16/13	22/21	33/28
112/99	14/11	40/39	15/13	52/45	13/10
44/39	33/26	104/99	13/11	56/51	21/17
68/63	17/14	64/57	24/19	19/18	19/16
256/243	32/27	9/8	81/64	52/51	39/34
136/117	17/13	7/6	21/16	64/63	8/7
80/68	30/23	56/45	23/20	24/23	27/23
92/81	23/18	184/171	57/46	76/69	23/19

x = 40/39, PI = 15/13

ENHARMONIC

40/39 · 39/38 · 19/15	ERATOSTHENES
40/39 · 26/25 · 5/4	AVICENNA

CHROMATIC

40/39 · 13/12 · 6/5	BARBOUR
40/39 · 39/35 · 7/6	
40/39 · 11/10 · 13/11	

DIATONIC

40/39 · 52/45 · 9/8	
40/39 · 91/80 · 8/7	

x = 28/27, PI = 7/6

ENHARMONIC

28/27 · 36/35 · 5/4	ARCHYTAS

CHROMATIC

28/27 · 243/224 · 32/27	ARCHYTAS
28/27 · 15/14 · 6/5	PTOLEMY
28/27 · 27/26 · 26/21	MAIN CATALOG

DIATONIC

28/27 · 8/7 · 9/8	ARCHYTAS
28/27 · 39/35 · 15/13	MAIN CATALOG

x = 16/15, PI = 6/5

CHROMATIC

16/15 · 25/24 · 6/5	DIDYMOS
16/15 · 15/14 · 7/6	AL-FARABI
16/15 · 20/19 · 19/16	KORNERUP

DIATONIC

16/15 · 9/8 · 10/9	PTOLEMY
16/15 · 13/12 · 15/13	MAIN CATALOG

Mean tetrachords

The mathematician and musician Archytas may have been the first to recognize the importance of the arithmetic, harmonic, and geometric means to music. He was credited with renaming the mean formerly called the "subcontrary" as the harmonic mean because it produced more pleasing melodic divisions than the arithmetic mean (Heath [1921] 1981; Erickson 1965). His own tunings were constructed by the application of only the harmonic and arithmetic means, but there were actually nine other means known to Greek mathematicians and which might be used to construct tetrachords (Heath [1921] 1981).

To this set of twelve may be added the *root mean square* or *quadratic mean* and four of my own invention whose definitions are given along with the historical ones in 4-4. The logarithmic mean divides an interval into two parts, the ratio of whose widths is the inverse of the ratio of the extremes of the interval. For example, the logarithmic mean divides the 2/1 into two

4-4. *Means: formulae and equivalent expressions from Heath 1921, 1:85–87, except for the logarithmic, ratio, and root mean square means. Number 12 is the framework of the scale when* a = 12 *and* b = 6. *The tetrachords generated by number 17 are extremely close numerically to the counter-logarithmic mean tetrachords of the other kinds. They also resemble the subcontraries to the geometric means.*

1. ARITHMETIC
$(a-b)/(b-c) = a/a = b/b = c/c \quad a+c = 2b$

2. GEOMETRIC
$(a-b)/(b-c) = a/b = b/c \quad ac = b^2$

3. HARMONIC
$(a-b)/(b-c) = a/c, 1/a + 1/c = 2/b \quad b = 2ac/(a+c)$

4. SUBCONTRARY TO HARMONIC
$(a-c)/(b-c) = c/a \quad (a^2+c^2)/(a+c) = b$

5. FIRST SUBCONTRARY TO GEOMETRIC
$(a-b)/(b-c) = c/b \quad a = b+c-c^2/b$

6. SECOND SUBCONTRARY TO GEOMETRIC
$(a-b)/(b-c) = b/a \quad c = a+b-a^2/b$

7. UNNAMED
$(a-c)/(b-c) = a/c \quad c^2 = 2ac - ab$

8. UNNAMED
$(a-c)/(a-b) = a/c \quad a^2+c^2 = a(b+c)$

9. UNNAMED
$(a-c)/(b-c) = b/c \quad b^2+c^2 = c(a+b)$

10. UNNAMED (SAME AS FIBONACCI SERIES)
$(a-c)/(a-b) = b/c \quad a = b+c$

11. UNNAMED
$(a-c)/(a-b) = a/b \quad a^2 = 2ab - bc$

12. MUSICAL PROPORTION
$a : (a+b)/2 = 2ab/(a+b) : b$

13. LOGARITHMIC MEAN
$\log b = (c \log a + a \log c)/(a+c) \quad (b/a)^c = (c/b)^a$

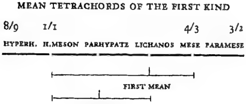
14. COUNTER-LOGARITHMIC MEAN
$\log b = (a \log a + c \log c)/(a+c) \quad (b/a)a = (c/b)c$

15. RATIO MEAN
$(a-c)/(b-c) = x/y \quad c = (bx-ay)/(x-y)$

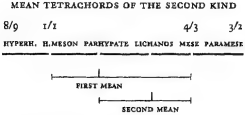
16. SECOND RATIO MEAN
$(a-c)/(a-b) = x/y \quad c = (ay-ax+bx)/y$

17. ROOT MEAN SQUARE
$b = \sqrt{((a^2+c^2)/2)} \quad b^2 = (a^2+c^2)/2$

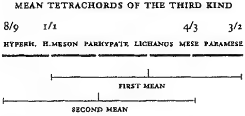
4-5. *Generating tetrachords with means.*



Lichanos is defined as the appropriate mean between hypate meson (1/1) and mese (4/3). Parhypate is then computed as the identical mean between lichanos and hypate.



Parhypate is defined as the appropriate mean between hypate meson (1/1) and mese (4/3). Lichanos is then computed as the identical mean between parhypate and mese.



Lichanos is defined as the appropriate mean between hypate meson (1/1) and paramese (3/2). Parhypate is then computed as the identical mean between mese (4/3) and hyperhypate (8/9).

intervals of 400 and 800 cents in the proportion of 1:2 (0, 400, and 1200 cents). The *counter-logarithmic mean* effects the same division in the opposite order, i.e., 800 and 400 cents (0, 800, and 1200 cents).

The two *ratio means*, numbers 15 and 16, are variations of numbers 7 and 8 of 4-4, differing only in that the ratio of the difference of the extremes to the difference between the mean and one of the extremes is dependent upon the parameter x/y.

There are still other types of mean, but these seventeen are sufficient to generate a considerable number of tetrachords (4-6–8) and may be of further utility in the algorithmic generation of melodies.

The most obvious procedures for generating tetrachords from these means are shown in 4-5. Mean tetrachords of the first kind are constructed by first calculating the lichanos as the mean between 1/1 and 4/3, or equivalently between $a = 4$ and and $c = 3$. The next step is the computation of parhypate as the same mean between 1/1 and the just calculated lichanos (4-6). Tetrachords of the second kind have the mean operations performed in reverse order (4-7). Tetrachords of the third kind are found by taking the means between 1/1 and 3/2 and between 8/9 and 4/3 (4-8); the smaller is defined as parhypate; the larger becomes the lichanos.

The construction of sets of genera analogous to those of Archytas, which are composed of a mean between 8/9 and 4/3 and its "subcontrary" or "counter"-mean between 8/9 and 32/27 (Erickson 1965; Winnington-Ingram 1932), is left for future investigations as it involves deep questions about the integration of intervals into musical systems.

Multiple means may be defined for the arithmetic, harmonic, and geometric means. The insertion of two arithmetic or harmonic means into the 4/3 results in Ptolemy's equable diatonic and its intervallic retrograde, 12/11 · 11/10 · 10/9, 10/9 · 11/10 · 12/11. The geometric mean equivalent is the new genus 166.667 + 166.667 + 166.667 cents (see the discussion of tempered tetrachords below).

4-6. *Mean tetrachords of the first kind. The lichanoi are the means between 1/1 and 4/3; the parhypatai are the means between 1/1 and the lichanoi.*

1. ARITHMETIC	1/1 13/12 7/6 4/3	13/12 · 14/13 · 8/7	139 + 128 + 231
2. GEOMETRIC	1.0 1.07457 1.15470 1.33333	1.07457 · 1.07457 · 1.15470	125 + 125 + 249
3. HARMONIC	1/1 16/15 8/7 4/3	16/15 · 15/14 · 7/6	112 + 119 + 267
4. SUBCONTRARY TO HARMONIC	1/1 533/483 25/21 4/3	533/483 · 575/533 · 28/25	171 + 131 + 196
5. FIRST SUBCONTRARY TO GEOMETRIC	1.0 1.09429 1.18046 1.33333	1.09429 · 1.07874 · 1.12950	156 + 131 + 211
6. SECOND SUBCONTRARY TO GEOMETRIC	1.0 1.09185 1.17704 1.33333	1.09185 · 1.07803 · 1.13278	152 + 130 + 216
7. UNNAMED	1/1 6/5 5/4 4/3	6/5 · 25/24 · 16/15	316 + 71 + 112
8. UNNAMED	1/1 157/156 13/12 4/3	157/156 · 169/157 · 16/13	11 + 128 + 359
9. UNNAMED	1.0 1.21677 1.26376 1.33333	1.21677 · 1.03862 · 1.05505	340 + 66 + 93
10. FIBONACCI SERIES	NO SOLUTION	—	—
11. UNNAMED	1/1 256/255 16/15 4/3	256/255 · 17/16 · 5/4	7 + 105 + 386
12. MUSICAL PROPORTION	1/1 8/7 7/6 4/3	8/7 · 49/48 · 8/7	231 + 36 + 231
13. LOGARITHMIC MEAN	1.0 1.05956 1.13122 1.33333	1.05956 · 1.06763 · 1.17867	100 + 113 + 285
14. COUNTER-LOGARITHMIC MEAN	1.0 1.09301 1.17867 1.33333	1.09301 · 1.07837 · 1.13122	154 + 131 + 213
15. RATIO MEAN (X/Y = 4/3)	1/1 19/16 5/4 4/3	19/16 · 20/19 · 16/15	298 + 89 + 112
16. SECOND RATIO MEAN (X/Y = 4/3)	1/1 157/156 13/12 4/3	157/156 · 169/157 · 16/13	11 + 128 + 359
17. ROOT MEAN SQUARE	1.0 1.09290 1.17851 1.33333	1.09291 · 1.078328 · 1.13137	154 + 131 + 214

4-7. *Mean tetrachords of the second kind. The parhypatai are the means between 1/1 and 4/3; the lichanoi are the means between the parhypatai and 4/3.*

1. ARITHMETIC	1/1 7/6 5/4 4/3	7/6 · 15/14 · 16/15	267 + 119 + 112
2. GEOMETRIC	1.0 1.15470 1.24081 1.33333	1.15470 · 1.07457 · 1.07457	249 + 125 + 125
3. HARMONIC	1/1 8/7 16/13 4/3	8/7 · 14/13 · 13/12	231 + 128 + 139
4. SUBCONTRARY TO HARMONIC	1/1 25/21 1409/1113 4/3	25/21 · 1409/1325 · 1484/1409	302 + 106 + 90
5. FIRST SUBCONTRARY TO GEOMETRIC	1.0 1.18046 1.25937 1.33333	1.18046 · 1.06685 · 1.05873	287 + 112 + 99
6. SECOND SUBCONTRARY TO GEOMETRIC	1.0 1.17704 1.25748 1.33333	1.17704 · 1.06833 · 1.06032	282 + 114 + 101
7. UNNAMED	1/1 5/4 85/64 4/3	5/4 · 17/16 · 256/255	386 + 105 + 7
8. UNNAMED	1/1 13/12 217/192 4/3	13/12 · 217/208 · 256/217	139 + 73 + 286
9. UNNAMED	1.0 1.26376 1.3299 1.33333	1.26376 · 1.05321 · 1.00260	405 + 88 + 4
10. FIBONACCI SERIES	NO SOLUTION	—	—
11. UNNAMED	1/1 16/15 10/9 4/3	16/15 · 25/24 · 6/5	112 + 71 + 316
12. MUSICAL PROPORTION	1/1 8/7 7/6 4/3	8/7 · 49/48 · 8/7	231 + 36 + 23
13. LOGARITHMIC MEAN	1.0 1.13122 1.21987 1.33333	1.13122 · 1.07837 · 1.09301	213 + 131 + 154
14. COUNTER-LOGARITHMIC MEAN	1.0 1.17867 1.25839 1.33333	1.17867 · 1.06763 · 1.05956	285 + 113 + 100
15. RATIO MEAN (X/Y=4/3)	1/1 5/4 21/16 4/3	5/4 · 21/20 · 64/63	386 + 84 + 27
16. RATIO MEAN (X/Y=4/3)	1/1 13/12 55/48 4/3	13/12 · 55/52 · 64/55	139 + 97 + 262
17. ROOT MEAN SQUARE	1.0 1.17851 1.22583 1.33333	1.17851 · 1.067708 · 1.059625	284 + 113 + 100

Summation tetrachords

Closely related to these applications of the various means is a simple nique which generates certain historically known tetrachords as v some unusual divisions. Wilson has called this *freshman sums*, and h plied it in many different musical contexts (Wilson 1974, 1986, 1989 numerators and denominators of two ratios are summed separat obtain a new fraction of intermediate size (Lloyd and Boyle 1978 example, the freshman sum of 1/1 and 4/3 is 5/4 , and the sum of 5 1/1 is 6/5. These ratios define the tetrachord 1/1 6/5 5/4 4/3. Similar "sum" of 5/4 and 4/3 is 9/7, and these ratios delineate the 1/1 5/4 9 tetrachord. The former is a permutation of Didymos's chromatic gen the latter is the inversion of Archytas's enharmonic. If one emp multiplier/index as in 4-2 and expresses the 1/1 as 2/2, 3/3. . . , an i set of graded tetrachords may be generated. The most important a teresting ones are tabulated in 4-9.

Similarly, the multiplier may be applied to the 4/3 rather than the yield 8/6, 12/9. . . . The resulting tetrachords fall into the enharmor hyperenharmonic classes and very quickly comprise intervals too sı be musically useful. A few of the earlier members are listed in 4-10.

4-8. *Mean tetrachords of the third kind. The lichanoi of these tetrachords are the means between 1/1 and 3/2; the parhypatai are the means between 8/9 and 4/3. These tetrachords are also tritriadic genera.*

1. ARITHMETIC	1/1 10/9 5/4 4/3	10/9 · 9/8 · 16/15	182 + 204 + 11
2. GEOMETRIC	1.0 1.08866 1.22474 1.33333	1.08866 · 1.125 · 1.08866	147 + 204 + 14
3. HARMONIC	1/1 16/15 6/5 4/3	16/15 · 9/8 · 10/9	112 + 204 + 18
4. SUBCONTRARY TO HARMONIC	1/1 52/45 13/12 4/3	52/45 · 9/8 · 40/39	250 + 204 + 44
5. FIRST SUBCONTRARY TO GEOMETRIC	1.0 1.13847 1.28078 1.33333	1.13847 · 1.125 · 1.0410	225 + 204 + 70
6. SECOND SUBCONTRARY TO GEOMETRIC	1.0 1.12950 1.27069 1.33333	1.1295 · 1.125 · 1.0493	211 + 204 + 83
7. UNNAMED	NO SOLUTION	—	—
8. UNNAMED	1/1 28/27 7/6 4/3	28/27 · 9/8 · 8/7	63 + 204 + 231
9. UNNAMED	NO SOLUTION	—	—
10. FIBONACCI SERIES	NO SOLUTION	—	—
11. UNNAMED	NO SOLUTION	—	—
12. MUSICAL PROPORTION	NOT DEFINED	—	—
13. LOGARITHMIC MEAN	1.0 1.04540 1.17608 1.33333	1.0454 · 1.125 · 1.1337	77 + 204 + 217
14. COUNTER-LOGARITHMIC MEAN	1.0 1.13371 1.27542 1.33333	1.1337 · 1.125 · 1.0454	217 + 204 + 77
15. RATIO MEAN (X/Y = 2/1)	1/1 10/9 5/4 4/3	10/9 · 9/8 · 16/15	182 + 204 + 11
16. RATIO MEAN (X/Y = 2/1)	1/1 10/9 5/4 4/3	10/9 · 9/8 · 16/15	182 + 204 + 11
17. ROOT MEAN SQUARE	1.0 1.1331 1.27475 1.33333	1.1331 · 1.125 · 1.04595	216 + 204 + 78

4-9. *Summation tetrachords of the first type. Unreduced ratios have been retained to clarify the generating process.*

	TETRACHORD	RATIOS	SOURCE
1.	1/1 6/5 5/4 4/3	6/5 · 25/24 · 16/15	DIDYMOS
2.	1/1 5/4 9/7 4/3	5/4 · 36/35 · 28/27	ARCHYTAS
3.	2/2 8/7 6/5 4/3	8/7 · 21/20 · 10/9	PTOLEMY
4.	2/2 6/5 10/8 4/3	6/5 · 25/24 · 16/15	DIDYMOS
5.	3/3 10/9 7/6 4/3	10/9 · 21/20 · 8/7	PTOLEMY
6.	3/3 7/6 11/9 4/3	7/6 · 22/21 · 12/11	PTOLEMY
7.	4/4 12/11 8/7 4/3	12/11 · 22/21 · 7/6	PTOLEMY
8.	4/4 8/7 12/10 4/3	8/7 · 21/20 · 10/9	PTOLEMY
9.	5/5 14/13 9/8 4/3	14/13 · 117/112 · 32/27	MISC. CAT.
10.	5/5 9/8 13/11 4/3	9/8 · 104/99 · 44/39	MAIN CAT.
11.	6/6 16/15 10/9 4/3	16/15 · 25/24 · 6/5	DIDYMOS
12.	6/6 10/9 14/12 4/3	10/9 · 21/20 · 7/6	PTOLEMY
13.	7/7 18/17 11/10 4/3	18/17 · 187/180 · 40/33	MISC. CAT.
14.	7/7 11/10 15/13 4/3	11/10 · 150/143 · 52/45	MISC. CAT.
15.	8/8 20/19 12/11 4/3	20/19 · 57/55 · 11/9	MAIN CAT.
16.	8/8 12/11 16/14 4/3	12/11 · 22/21 · 7/6	PTOLEMY
17.	9/9 22/21 13/12 4/3	22/21 · 91/88 · 16/13	MISC. CAT.
18.	9/9 13/12 17/15 4/3	13/12 · 68/65 · 20/17	MAIN CAT.
19.	10/10 24/23 14/13 4/3	24/23 · 161/156 · 26/21	MISC. CAT.
20.	10/10 14/13 18/16 4/3	14/13 · 117/112 · 32/27	MISC. CAT.
21.	11/11 26/25 15/14 4/3	26/25 · 375/364 · 56/45	MISC. CAT.
22.	11/11 15/14 19/17 4/3	15/14 · 266/255 · 68/57	MISC. CAT.
23.	12/12 28/27 16/15 4/3	28/27 · 36/35 · 5/4	ARCHYTAS
24.	12/12 16/15 20/18 4/3	16/15 · 25/24 · 6/5	DIDYMOS

4-10. *Summation tetrachords of the second type. Unreduced ratios have been retained to clarify the generating process.*

	TETRACHORD	RATIOS	SOURCE
1.	1/1 10/8 9/7 8/6	5/4 · 36/35 · 28/27	ARCHYTAS
2.	1/1 9/7 17/13 8/6	9/7 · 119/117 · 52/51	MISC. CAT.
3.	1/1 14/11 13/10 12/9	14/11 · 143/140 · 40/39	MISC. CAT.
4.	1/1 13/10 25/19 12/9	13/10 · 250/247 · 76/75	MISC. CAT.
5.	1/1 18/14 17/13 16/12	9/7 · 119/117 · 52/51	MISC. CAT.
6.	1/1 17/13 33/25 16/12	17/13 · 429/425 · 100/99	MISC. CAT.
7.	1/1 22/17 21/16 20/15	22/17 · 357/352 · 64/63	MISC. CAT.
8.	1/1 21/16 41/31 20/15	21/16 · 656/651 · 124/123	MISC. CAT.
9.	1/1 26/20 25/19 24/18	13/10 · 250/247 · 76/75	MISC. CAT.
10.	1/1 25/19 49/37 24/18	25/19 · 931/925 · 148/147	MISC. CAT.

4-11. *Neo-Aristoxenian genera with constant CI.*

PARTS	CENTS	APPROXIMATION	PTOLEMAIC INTERPRETA
ENHARMONIC			
1.5 + 1.5 + 27	25 + 25 + 450	80/79 · 79/78 · 13/10	80/79 · 79/78 · 13/10
1 + 2 + 27	17 + 33 + 450	120/119 · 119/117 · 13/10	120/119 · 119/117 · 13/1
2 + 2 + 26	33 + 33 + 433	56/55 · 55/54 · 9/7	60/59 · 59/58 · 58/45
2.5 + 2.5 + 25	42 + 42 + 417	44/43 · 43/42 · 14/11	48/47 · 47/46 · 23/18
2 + 3 + 25	33 + 50 + 417	55/54 · 36/35 · 14/11	60/59 · 118/115 · 23/18
2 + 4 + 24	33 + 67 + 400	60/59 · 59/57 · 19/15	60/59 · 59/57 · 19/15
3 + 3 + 24	50 + 50 + 400	40/39 · 39/38 · 19/15	40/39 · 38/39 · 19/15
2 + 5 + 23	33 + 83 + 383	56/55 · 22/21 · 5/4	60/59 · 118/113 · 113/9
3 + 4 + 23	50 + 67 + 383	36/35 · 28/27 · 5/4	40/39 · 117/113 · 113/9
3.5 + 3.5 + 23	58 + 58 + 383	32/31 · 31/30 · 5/4	240/233 · 233/226 · 113/
CHROMATIC			
2 + 6 + 22	33 + 100 + 367	51/50 · 18/17 · 100/81	60/59 · 59/56 · 56/45
8/3 + 16/3 + 22	44 + 89 + 367	40/39 · 21/20 · 26/21	45/44 · 22/21 · 56/45
3 + 5 + 22	50 + 83 + 367	34/33 · 22/21 · 21/17	40/39 · 117/112 · 56/45
4 + 4 + 22	67 + 67 + 367	28/27 · 27/26 · 26/21	30/29 · 29/28 · 56/45
2 + 7 + 21	33 + 117 + 350	56/55 · 15/14 · 11/9	60/59 · 118/111 · 37/30
3 + 6 + 21	50 + 100 + 350	34/33 · 18/17 · 11/9	40/39 · 39/37 · 37/30
4 + 5 + 21	67 + 83 + 350	28/27 · 22/21 · 27/22	30/29 · 116/111 · 37/30
4.5 + 4.5 + 21	75 + 75 + 350	24/23 · 23/22 · 11/9	80/77 · 77/74 · 37/30
2 + 10 + 18	33 + 167 + 300	45/44 · 11/10 · 32/27	60/59 · 59/54 · 6/5
3 + 9 + 18	50 + 150 + 300	33/32 · 12/11 · 32/27	40/39 · 13/12 · 6/5
4 + 8 + 18	67 + 133 + 300	28/27 · 243/224 · 32/27	30/29 · 29/27 · 6/5
4.5 + 7.5 + 18	75 + 125 + 300	25/24 · 27/25 · 32/27	80/77 · 77/72 · 6/5
5 + 7 + 18	83 + 117 + 300	21/20 · 15/14 · 32/27	24/23 · 115/108 · 6/5
6 + 6 + 18	100 + 100 + 300	256/243 · 2187/2048 · 32/27	20/19 · 19/18 · 6/5
DIATONIC			
2 + 13 + 15	33 + 217 + 250	45/44 · 44/39 · 52/45	60/59 · 118/105 · 7/6
3 + 12 + 15	50 + 200 + 250	34/33 · 19/17 · 22/19	40/39 · 39/35 · 7/6
4 + 11 + 15	67 + 183 + 250	27/26 · 10/9 · 52/45	30/29 · 116/105 · 7/6
5 + 10 + 15	83 + 167 + 250	104/99 · 11/10 · 15/13	24/23 · 23/21 · 7/6
6 + 9 + 15	100 + 217 + 250	19/18 · 12/119 · 22/19	20/19 · 38/35 · 7/6
7 + 8 + 15	117 + 217 + 250	104/97 · 97/909 · 15/13	120/113 · 113/105 · 7/6
7.5 + 7.5 + 15	125 125 + 250	15/14 · 14/13 · 52/45	16/15 · 15/14 · 7/6
2 + 16 + 12	33 + 267 + 200	64/63 · 7/6 · 9/8	60/59 · 59/51 · 17/15
3 + 15 + 12	50 + 250 + 200	40/39 · 52/45 · 9/8	40/39 · 39/34 · 17/15
4 + 14 + 12	67 + 233 + 200	28/27 · 8/7 · 9/8	30/29 · 58/51 · 17/15
4.5 + 13.5 + 12	75 + 225 + 200	24/23 · 92/81 · 9/8	80/77 · 77/68 · 17/15
5 + 13 + 12	83 + 217 + 200	22/21 · 112/90 · 9/8	24/23 · 115/102 · 17/15
6 + 12 + 12	100 + 200 + 200	256/243 · 9/8 · 9/8	20/19 · 19/17 · 17/15
7 + 11 + 12	117 + 183 + 200	16/15 · 10/9 · 9/8	120/113 · 113/102 · 17/1
8 + 10 + 12	113 + 167 + 200	320/297 · 11/10 · 9/8	15/14 · 56/51 · 17/15

Neo-Aristoxenian tetrachords with Ptolemaic interpretations

While Aristoxenos may have been documenting contemporary practice, even a cursory look at his tables suggests that many plausible neo-Aristoxenian genera could be constructed to "fill in the gaps" in his set. The most obvious missing genera are a diatonic with enharmonic diesis, 3 + 15 + 12 (50 + 250 + 200 cents), a *parachromatic*, 5 + 5 + 20 (83 + 83 + 334 cents), and a new soft diatonic, 7.5 + 7.5 + 15 (125 + 125 + 250 cents).

Although Aristoxenos favored genera with 1:1 divisions of the pyknon, Ptolemy and the Islamic writers preferred the 1:2 relation. More complex divisions, of course, are also possible. 4-11 lists a number of neo-Aristoxenian genera in which the CI is held constant and the pyknotic division is varied. With the exception of the first five genera which represent *hyperenharmonic* forms and three which are a closer approximation of the enharmonic (383 cents, rather than 400 cents), only Aristoxenos's CIs are used.

For each tempered genus an approximation in just intonation is selected from a genus in the Main Catalog. Furthermore, an approximation in terms of fractional parts of a string of 120 units of length, analogous to Ptolemy's interpretation of Aristoxenos's genera, is also provided. While these *Ptolemaic interpretations* are occasionally quite close to the ideal tempered forms, they often deviate substantially. One should note, however, that the Ptolemaic approximations are more accurate for the smaller intervals than the larger.

Intervals whose sizes fall between one third and one half of the perfect fourth may be be repeated within the tetrachord, leaving a remainder less than themselves. These are termed reduplicated genera and a representative set of such neo-Aristoxenian tetrachords with reduplication is shown in 4-12.

4-12. *Neo-Aristoxenian genera with reduplication.*

PARTS	CENTS	APPROXIMATION	PTOLEMAIC INTERPRETATION
2 + 14 + 14	34 + 233 + 233	49/48 · 8/7 · 8/7	60/59 · 59/52 · 52/45
4 + 13 + 13	67 + 217 + 217	300/289 · 17/15 · 17/15	30/29 · 116/103 · 103/90
6 + 12 + 12	100 + 200 + 200	256/243 · 9/8 · 9/8	20/19 · 19/17 · 17/15
8 + 11 + 11	133 + 183 + 183	27/25 · 10/9 · 10/9	15/14 · 112/101 · 101/90
10 + 10 + 10	166 + 167 + 167	11/10 · 11/10 · 400/363	12/11 · 11/10 · 10/9

4-13. *Neo-Aristoxenian genera with constant pyknotic proportions.*

1:1 PYKNON	CENTS	APPROXIMATION	PTOLEMAIC INTERPRETAT
1.5 + 1.5 + 27	25 + 25 + 450	80/79 · 79/78 · 13/10	80/79 · 79/78 · 13/10
2 + 2 + 26	33 + 33 + 433	56/55 · 55/54 · 9/7	60/59 · 59/58 · 58/45
2.5 + 2.5 + 25	42 + 42 + 417	44/43 · 43/42 · 14/11	48/47 · 47/46 · 23/18
3 + 3 + 24	50 + 50 + 400	40/39 · 39/38 · 19/15	40/39 · 39/38 · 19/15
3.5 + 3.5 + 23	58 + 58 + 383	32/31 · 31/30 · 5/4	240/233 · 233/226 · 113/
4 + 4 + 22	67 + 67 + 367	28/27 · 27/26 · 26/21	30/29 · 29/28 · 56/45
4.5 + 4.5 + 21	75 + 75 + 350	24/23 · 23/22 · 11/9	80/77 · 77/74 · 37/30
5 + 5 + 20	83 + 83 + 334	22/21 · 21/20 · 40/33	24/23 · 23/22 · 11/9
5.5 + 5.5 + 19	92 + 92 + 317	20/19 · 19/18 · 6/5	240/229 · 229/218 · 109/
6 + 6 + 18	100 + 100 + 300	18/17 · 17/16 · 32/27	20/19 · 19/18 · 6/5
6.5 + 6.5 + 17	108 + 108 + 283	17/16 · 16/15 · 20/17	240/227 · 227/214 · 107/
7 + 7 + 16	117 + 117 + 267	16/15 · 15/14 · 7/6	120/113 · 113/106 · 53/4
7.5 + 7.5 + 15	125 + 125 + 250	15/14 · 14/13 · 52/45	16/15 · 15/14 · 7/6
8 + 8 + 14	133 + 133 + 234	14/13 · 13/12 · 7/6	15/14 · 14/13 · 52/45
8.5 + 8.5 + 13	142 + 142 + 217	40/37 · 37/34 · 17/15	240/223 · 223/206 · 103/
9 + 9 + 12	150 + 150 + 200	64/59 · 59/54 · 9/8	40/37 · 37/34 · 17/15
9.5 + 9.5 + 11	158 + 158 + 183	12/11 · 11/10 · 10/9	240/221 · 221/202 · 101/
10 + 10 + 10	166 + 166 + 167	11/10 · 11/10 · 400/363	12/11 · 11/10 · 10/9
1:2 PYKNON			
1 + 2 + 27	17 + 33 + 450	120/119 · 119/117 · 13/10	120/119 · 119/117 · 13/1
4/3 + 8/3 + 26	22 + 44 + 433	84/83 · 83/81 · 9/7	90/89 · 89/87 · 58/45
5/3 + 10/3 + 25	28 + 56 + 417	64/63 · 33/32 · 14/11	72/71 · 71/69 · 23/18
2 + 4 + 24	33 + 67 + 400	57/56 · 28/27 · 24/19	60/59 · 59/57 · 19/15
7/3 + 14/3 + 23	39 + 78 + 383	46/45 · 24/23 · 5/4	360/353 · 353/339 · 113/
8/3 + 16/3 + 22	44 + 89 + 367	40/39 · 21/20 · 26/21	45/44 · 22/21 · 56/45
3 + 6 + 21	50 + 100 + 350	34/33 · 18/17 · 11/9	40/39 · 39/37 · 37/30
10/3 + 20/3 + 20	56 + 111 + 333	33/32 · 16/15 · 40/33	36/35 · 35/33 · 11/9
11/3 + 22/3 + 19	61 + 122 + 317	28/27 · 15/14 · 6/5	360/349 · 349/327 · 109/
4 + 8 + 18	67 + 133 + 300	27/26 · 13/12 · 32/27	30/29 · 29/27 · 6/5
13/3 + 26/3 + 17	72 + 144 + 283	51/49 · 49/45 · 20/17	360/347 · 347/321 · 107/
14/3 + 28/3 + 16	78 + 156 + 267	22/21 · 12/11 · 7/6	180/173 · 173/159 · 53/4
5 + 10 + 15	83 + 167 + 250	104/99 · 11/10 · 15/13	24/23 · 23/22 · 7/6
16/3 + 32/3 + 14	89 + 178 + 233	21/20 · 10/9 · 8/7	45/43 · 43/39 · 52/45
17/3 + 34/3 + 13	94 + 189 + 217	20/19 · 19/17 · 20/17	360/343 · 343/309 · 103/
6 + 12 + 12	100 + 200 + 200	256/243 · 9/8 · 9/8	20/19 · 19/17 · 17/15

Finally, in 4-13, the pyknotic proportions are kept constant at either 1:1 or 1:2 and the CIs are allowed to vary.

These neo-Aristoxenian tetrachords may be approximated in just intonation or realized in equal temperaments whose cardinalities are zero modulo 12. The zero modulo 12 temperaments provide opportunities to simulate many of the other genera in the Catalogs as their fourths are only two cents from 4/3 and other intervals of just intonation are often closely approximated. One may also use them to discover or invent new neo-Aristoxenian tetrachords.

To articulate a single part difference, a temperament of 72 tones per octave is required. The 1/2 parts in the hemiolic chromatic and several other genera normally demand 144 tones unless all the intervals including the disjunctive tone have a common factor. In this case, the 48-tone system suffices. For the 1:2 pykna which employ 1/3 parts, 216-tone temperament is necessary unless the numbers of parts share common factors. These data are summarized in 4-14.

4-14. *Aristoxenian realizations. The framework is the number of "parts" in the two tetrachords and the disjunctive tone. The corresponding equal temperament is the sum of the parts of the framework. The articulated genera are those that may be played in the corresponding equal temperaments. The scheme of 144 parts was used by Avicenna and Al-Farabi (D'Erlanger 1930).*

FRAMEWORK	ET	ARTICULATED GENERA
5 2 5	12	*Diatonic and syntonic chromatic.*
10 4 10	24	*Enharmonic, syntonic and soft diatonics, syntonic chromatic.*
15 6 15	36	*Syntonic diatonic, syntonic and soft chromatics, unnamed. Chromatic, diatonic with soft chromatic dieses.*
20 8 20	48	*Hemiolic chromatic, soft and syntonic diatonics, syntonic chromatic, diatonic with hemiolic chromatic dieses. See 24-tone ET.*
25 10 25	60	*Syntonic diatonic and chromatic.*
30 12 30	72	*All previous genera except hemiolic chromatic and genera with hemiolic chromatic dieses (see 24-tone ET).*
35 14 35	84	*Syntonic diatonic and chromatic.*
40 16 40	96	*Enharmonic, syntonic diatonic, soft diatonic, syntonic and hemiolic chromatic. See 24-tone ET.*
45 18 45	108	*See 36-tone ET.*
50 20 50	120	*See 24-tone ET.*
55 22 55	132	*See 12-tone ET.*
60 24 60	144	*All genera except 1:2 pykna with 1/3 parts.*
90 36 90	216	*All genera defined in text.*

Semi-tempered tetrachords

The computation of the mean tetrachords also generates a number of genera containing irrational intervals involving square roots. These tetrachords contain both tempered intervals as well as at least one in just intonation, the 4/3, and may therefore be called *semi-tempered.* There also are the semi-tempered tetrachords resulting from a literal interpretation of the late classical theorists Nichomachos and Thrasyllus (Barbera 1978). The first of these is Nichomachos's enharmonic, defined verbally as a ditone with an equally divided *limma* and mathematically as $\sqrt{(256/243)} \cdot \sqrt{(256/243)} \cdot 81/64$ (45 + 45 + 408 cents). The second is Thrasyllus's chromatic, described analogously as having a Pythagorean *trihemitone* or minor third and a whole tone pyknon. Literally, this genus would be $\sqrt{(9/8)} \cdot \sqrt{(9/8)} \cdot 32/27$ (102 + 102 + 294 cents), but it is possible that Thrasyllus meant the standard Pythagorean tuning in which the pyknon consists of a limma plus an *apotome,* i.e., $256/243 \cdot 2187/2048 \cdot 32/27$ (90 + 114 + 294 cents).

Other semi-tempered forms result from Barbera's assumption that Aristoxenos may have intended that the perfect fourth of ratio 4/3 be divided geometrically into thirty parts. Barbera (1978) offers this literal version of the enharmonic: $\sqrt[10]{(4/3)} \cdot \sqrt[10]{(4/3)} \cdot \sqrt[10]{(65536/6561)}$, or 50 + 50 + 398 cents, where 65536/6561 is $(4/3)^8$. It is an easy problem to find analogous interpretations of the remainder of Aristoxenos's genera. These and a few closely related genera from 3-1–3 have been tabulated in 4-15.

4-15. *Semi-tempered Aristoxenian tetrachords. These tetrachords are literal interpretations of Aristoxenos's genera under Barbera's assumption that Aristoxenos meant to divide the perfect fourth of ratio 4/3 into 30 equal parts.*

	PARTS	ROOTS	CENTS	GENUS
1.	3 + 3 + 24	$4/3^{1/10} \cdot 4/3^{1/10} \cdot 4/3^{4/5}$	50 + 50 + 398	ENHARMONIC
2.	4 + 4 + 22	$4/3^{2/15} \cdot 4/3^{2/15} \cdot 4/3^{11/15}$	66 + 66 + 365	SOFT CHROMATIC
3.	4.5 + 4.5 + 21	$4/3^{3/20} \cdot 4/3^{3/20} \cdot 4/3^{7/10}$	75 + 75 + 349	HEMIOLIC CHROMATIC
4.	6 + 6 + 18	$4/3^{1/5} \cdot 4/3^{1/5} \cdot 4/3^{3/5}$	100 + 100 + 299	INTENSE CHROMATIC
5.	6 + 9 + 15	$4/3^{1/5} \cdot 4/3^{3/10} \cdot 4/3^{1/2}$	100 + 149 + 250	SOFT DIATONIC
6.	6 + 12 + 12	$4/3^{1/5} \cdot 4/3^{2/5} \cdot 4/3^{2/5}$	100 + 199 + 199	INTENSE DIATONIC
7.	4 + 14 + 12	$4/3^{2/15} \cdot 4/3^{7/15} \cdot 4/3^{2/5}$	66 + 232 + 199	DIATONIC WITH SOFT CHROMATIC DIESES
8.	4.5 + 13.5 + 12	$4/3^{3/20} \cdot 4/3^{9/20} \cdot 4/3^{2/5}$	75 + 224 + 199	DIATONIC WITH HEMIOLIC CHROMATIC DIESES
9.	4 + 8 + 18	$4/3^{2/15} \cdot 4/3^{4/15} \cdot 4/3^{3/5}$	66 + 133 + 299	UNNAMED
10.	6 + 3 + 21	$4/3^{1/5} \cdot 4/3^{1/10} \cdot 4/3^{7/10}$	100 + 50 + 349	REJECTED
11.	4.5 + 3.5 + 22	$4/3^{3/20} \cdot 4/3^{7/60} \cdot 4/3^{11/15}$	75 + 58 + 365	REJECTED
12.	10 + 10 + 10	$4/3^{1/3} \cdot 4/3^{1/3} \cdot 4/3^{1/3}$	166 + 166 + 166	SEMI-TEMPERED EQUABLE DIATONIC
13.	12 + 9 + 9	$4/3^{2/5} \cdot 4/3^{3/10} \cdot 4/3^{3/10}$	200 + 149 + 149	ISLAMIC DIATONIC

Equal divisions of the 4/3

The semi-tempered tetrachords suggest that equally tempered divisions of the 4/3 would be worth exploring. Such scales would be analogous to the equal temperaments of the octave except that the interval of equivalence is the 4/3 rather than the 2/1. Scales of this type are very rare, though they have been reported to exist in contemporary Greek Orthodox liturgical music (Xenakis 1971).

A possible ancestor of such scales is the ancient Lesser Perfect System, which consisted of a chain of the three tetrachords hypaton, meson, and synemmenon. In theory, all three tetrachords were identical, but this was not an absolute requirement, and in fact, in Ptolemy's mixed tunings, they would not have been the same. (See chapter 6 for the derivations of the various scales and systems, and chapter 5 for the analysis of their properties.)

The most interesting equal divisions of the 4/3 resemble the equal temperaments described in the next section and in 4-14 and 4-17. The melodic possibilities of these scales should be quite rich, because in those divisions with more than three degrees to the 4/3 not only can several tetrachordal genera be constructed, but various permutations of these genera are also possible.

The harmonic properties, however, may be very different from those of the octave divisions as the 2/1 may not be approximated closely enough for octave equivalence to be retained. Moreover, depending upon the division, other intervals such as the 3/2 or 3/1 may or may not be acceptably consonant.

The equal divisions of the 4/3 which correspond to equal octaval temperaments are described in 4-16. A few supplementary divisions such as the one of 11 degrees have been added since they reasonably approximate harmonically important intervals. For reasons of space, only a very limited number of intervals was examined and tabulated. To gain an adequate understanding of these tunings, the whole gamut should be examined over a span of at least eight 4/3's.

Additionally, the nearest approximations to the octave and the number of degrees per 2/1 are listed. This information allows one to decide whether the tuning is equivalent to an octave division, or whether it essentially lacks octave equivalence. Composition in scales without octave equivalence is a relatively unexplored area, although the

DEGREES PER 4/3	CENTS/DEGREE	DEGREES/OCTAVE	CENTS/OCTAVE	OCTAVE DIVISION	OTHER CONSONANT INTERVA
3	166.0	7.228	1162.1	7 (–)	GOLDEN RATIO (PHI) = 5
4	124.5	9.638	1245.1	10 (+)	7/1 = 27
5	99.61	12.05	1195.3	12 (–)	5/1 = 28
6	83.01	14.46	1162.1	14 (–)	7/5 = 7
7	71.15	16.86	1209.5	17 (+)	—
8	62.26	19.27	1182.9	19 (–)	7/1 = 54
9	55.34	21.68	1217.4	22 (+)	5/3 = 16, 6/1 = 56
10	49.80	24.09	1195.3	24 (–)	3/2 = 14, 5/1 = 56
11	45.28	26.50	1222.5	27 (+)	3/1 = 42, 4/1 = 53, 5/2 = 35,
13	38.31	31.32	1187.6	31 (–)	6/1 = 81, 7/1 = 88, 8/1 = 94
14	35.57	33.73	1209.5	34 (+)	7/2 = 61
15	33.20	36.14	1195.3	36 (–)	5/1 = 84, PHI = 25
17	29.30	40.96	1201.2	41 (+)	3/2 = 24, 7/2 = 74
20	24.90	48.19	1195.3	48 (–)	5/1 = 112, 7/4 = 39
22	22.64	53.01	1199.8	53 (–)	3/2 = 31, 5/3 = 39
25	19.92	60.24	1195.3	60 (–)	5/1 = 140, 7/1 = 169
28	17.79	67.46	1191.8	67 (–)	3/1 = 107, 4/1 = 135
30	16.605	72.28	1195.3	72 (–)	7/1 = 203, 7/5 = 35
35	14.23	84.33	1195.3	84 (–)	7/4 = 68, 7/5 = 41
40	12.45	96.38	1195.3	96 (–)	6/1 = 249, 5/3 = 71
45	11.07	108.4	1195.3	108 (–)	3/1 = 172, 4/1 = 217
50	9.961	120.5	1195.3	120 (–)	3/1 = 191, 4/1 = 241
55	9.055	132.5	1204.4	133 (+)	7/4 = 107, PHI = 92, 3/1 = 21
60	8.301	144.6	1203.6	145 (+)	3/1 = 229, 4/1 = 289
90	5.534	216.8	1200.8	217 (+)	3/2 = 127

4-16. *Equal divisions of the 4/3. These are equal temperaments of the 4/3 rather than the 2/1. "Degrees/octave" is the number of degrees of the division corresponding to the 2/1 or octave. For many of these divisions, the octave no longer functions as an interval of equivalence. "Cents/octave" is the cent value of the approximations to the 2/1. "Octave division" is the closest whole number of degrees to the 2/1. (–) indicates that the octave is compressed and less than 1200 cents. (+) means that it is stretched and larger than 1200 cents. "Consonant intervals" are the degrees in good approximations to the intervals listed. All divisions of the 4/3 have good approximations to the 10/1 as $(4/3)^8$ + the skhisma equals 10/1. Divisons that are multiples of 3 also have good approximations to the 11/1. 17 is a slightly stretched 41-tone equal temperament. 22 is audibly equivalent to 53-tone equal temperament. 28 is analogous to the division of the fourth into 28 parts according to Tiby's theory of Greek Orthodox liturgical music (Tiby 1938). 30 is analogous to Aristoxenos's basic system. 55 is analogous to 132-tone equal temperament. 60 is analogous to 144-tone equal temperament. 90 is analogous to 216-tone equal temperament. The Golden Ratio or Phi is $(1+\sqrt{5})/2$, approximately 1.618.*

composer and theorist Brian McLaren has recently written a number of pieces in non-octaval scales mostly of his own invention (McLaren, personal communication, 1991). Xenakis has also mentioned chains of fifths consisting of tetrachords and disjunctive tones (Xenakis 1971). These suggest analogous divisions of the 3/2, including both those with good approximations to the 4/3 and those without. Similarly, there are divisions in which octave equivalence is retained and those in which it is not. An example of one with both good fourths and octaves is the seventh root of 3/2, which corresponds to a moderately stretched 12-tone equal temperament of the octave (Kolinsky 1959).

Tetrachords in non-zero modulo 12 equal temperaments

Tetrachords may also be defined in non-zero modulo 12 equal temperaments. For some combinations of genus and tuning the melodic and harmonic distortions will be negligible, but for others the mappings may distort the characteristic melodic shapes unacceptably. As an illustration, the three primary genera, the enharmonic, the syntonic chromatic, and the

4-17. *Tetrachords in non-zero modulo 12 equal temperaments. These genera are defined in ETs where the perfect fourth does not equal 2 1/2 "whole tones." The framework is the number of "parts" in the two fourths and the disjunctive tone. More than one framework is plausible in some temperaments without good fourths or with more than 17 notes. The corresponding equal temperament is the sum of the parts of the framework. The genera in a generalized, non-specific sense may be approximated in these equal temperaments. "Diatonic/chromatic" means that there is no melodic distinction between these genera. The chromatic pykna in 9-, 10-, and 11-tone ET consist of two small intervals and one large, while the disjunction may larger or smaller than the CI. Genera indifferently enharmonic and chromatic occur around 19 tones per octave and neo-Aristoxenian forms may be realizable in many of the ETs.*

FRAMEWORK	ET	GENERA
3 1 3	7	DIATONIC/CHROMATIC
3 2 3	8	DIATONIC/CHROMATIC
4 1 4	9	CHROMATIC
4 2 4	10	CHROMATIC
4 3 4	11	CHROMATIC
5 3 5	13	DIATONIC, CHROMATIC
6 2 6	14	DIATONIC, CHROMATIC
6 3 6	15	DIATONIC, CHROMATIC
7 2 7	16	DIATONIC, CHROMATIC
7 3 7	17	DIATONIC, CHROMATIC
7 4 7 (8 2 8)	18	DIATONIC, CHROMATIC (ALL THREE)
8 3 8	19	DIATONIC, CHROMATIC
8 4 8	20	ALL THREE
9 3 9, 8 5 8	21	ALL THREE
9 4 9	22	ALL THREE
9 5 9, 10 3 10	23	ALL THREE
13 5 13	31	ALL THREE
14 6 14	34	ALL THREE
17 7 17	41	ALL THREE
22 9 22	53	ALL THREE

4-18. *Augmented and diminished tetrachords. These tetrachords are closely related to those in 8-5 and 8-15. For tetrachords with perfect fourths incorporating the diminished fourths as intervals, see the Main and Miscellaneous Catalogs. A few additional intervals of similar size have been used as CIs in 4-1, but not divided due to their complexity. The last three intervals are technically diminished fifths, but they function as augmented fourths in certain of the harmoniai of chapter 8.*

RATIOS	CENTS	EXAMPLES
14/11	418	14/13 · 13/12 · 12/11
23/18	424	23/22 · 11/10 · 10/9
32/25	427	32/31 · 31/30 · 6/5
9/7	435	18/17 · 17/16 · 8/7
31/24	443	31/30 · 10/9 · 9/8
22/17	446	11/10 · 10/9 · 18/17
13/10	454	13/12 · 12/11 · 11/10
30/23	460	15/14 · 7/6 · 24/23
17/13	464	17/16 · 8/7 · 14/13
21/16	471	21/20 · 10/9 · 9/8
29/22	478	29/28 · 7/6 · 12/11
31/23	517	31/30 · 5/4 · 24/23
23/17	523	23/22 · 11/9 · 18/17
19/14	529	19/18 · 6/5 · 15/14
15/11	537	15/14 · 7/6 · 12/11
26/19	543	26/25 · 5/4 · 20/19
11/8	551	11/10 · 10/9 · 9/8
40/29	557	8/7 · 7/6 · 30/29
18/13	563	9/8 · 8/7 · 14/13
25/18	569	5/4 · 20/19 · 19/18
32/23	572	16/15 · 5/4 · 24/23
7/5	583	14/13 · 13/12 · 6/5
1024/729	588	256/243 · 8/7 · 7/6
45/32	590	16/15 · 10/9 · 6/5
24/17	597	6/5 · 10/9 · 18/17
17/12	603	17/16 · 8/7 · 7/6
44/31	606	11/10 · 5/4 · 32/31
10/7	617	10/9 · 9/8 · 8/7

diatonic, will be mapped into the 12-, 19-, 22-, and 24-tone equal temperament (ET) below:

ET	FOURTH	ENHARMONIC	CHROMATIC	DIATONIC
12	5°	—	1+1+3	1+2+2
19	8°	1+1+6	2+2+4	2+3+3
22	9°	1+1+7	2+2+5	1+4+4
24	10°	1+1+8	2+2+6	2+4+4

The enharmonic is not articulated in 12-tone ET, or at least not distinguishable from the chromatic except as a semitonal-major third pentatonic. In 19-tone ET, the soft chromatic is identical to the enharmonic and the syntonic chromatic is close to a diatonic genus like 125 + 125 + 250 cents. The enharmonic is certainly usable in 22-tone ET but the diatonic is deformed, with a quarter-tone taking the place of the semitone. These distortions, however, are mild compared to the 9-tone equal temperament in which not only are the diatonic and chromatic genera equivalent as 1 + 1 + 2 degrees, but the semitone at two units is larger than the whole tone. Whether these intervallic transmogrifications are musically useful remains to be tested.

There are, however, many fascinating musical resources in these non-12-tone tunings. As Ivor Darreg has pointed out, each of the equal temperaments has its own particular mood which suffuses any scale mapped into it (Darreg 1975). For this reason the effects resulting from transferring between tuning systems may be of considerable interest.

Because of the large number of systems to be covered, the mappings of the primary tetrachordal genera into the non-zero modulo 12 equal temperaments are summarized in 4-17. The tetrachordal framework and primary articulated genera in the equal temperaments of low cardinality or which are reasonable approximations to just intonation are shown in this figure.

Augmented and diminished tetrachords

The modified or altered tetrachords found in some of the non-zero modulo 12 equal temperaments of 4-17 suggest that tetrachords based on augmented and diminished fourths might be musically interesting. This supposition has historical and theoretical support. The basic scales (*thats*) of some Indian ragas have both augmented and perfect fourths (Sachs 1943), and the octaval *harmoniai* of Kathleen Schlesinger contain fourths of di-

magnitudes (Schlesinger 1939; and chapter 8). Wilson has exploited the fact that any scale generable by a chain of melodic fourths must incorporate fourths of at least two magnitudes (Wilson 1986; 1987; and chapter 6). His work implies that scales may be produced from chains of fourths of any type, but that their sizes and order must be carefully selected to ensure that the resulting scales are recognizably tetrachordal.

A number of altered fourths are available for experimentation. 4-18 lists those which commonly arise in conventional theory and in the extended theory of Schlesinger's harmoniai described in chapter 8. Scales may be constructed by combining these tetrachords with each other or with normal ones and with correspondingly altered disjunctive tones to complete the octaves. Alternatively, the methods described in chapter 6 to generate non-heptatonic scales may be employed.

5 Classification, characterization, and analysis of tetrachords

This chapter contains a complex mixture of topics regarding the description or characterization of tetrachords. Some of the concepts are chiefly applicable to single tetrachords, while others refer to pairs of tetrachords or the complete tetrachordal space. The most interesting of the newer methods, those of Rothenberg and Polansky, are most usefully applied to the scales and scale-like aggregates described in detail in chapter 6. Moreover, Polansky's methods may be applied to parameters other than pitch height. The application of these techniques to tetrachords may serve as an model for their use in broader areas of experimental intonation.

The first part of the chapter is concerned with the historical approach to classification and with two analyses based on traditional concepts. These concepts include classification by the size of the largest, and usually uppermost, incomposite interval and subclassification by the relative sizes of the two smallest intervals. A new and somewhat more refined classification scheme based on these historical concepts is proposed at the end of this section.

These concepts and relationships are displayed graphically in order that they may become more intuitively understood. A thorough understanding of the melodic properties of tetrachords is a prerequsite for effective composition with tetrachordally derived scales. Of particular interest are those tetrachords which lie near the border of two categories. Depending upon their treatment, they may be perceived as belonging to either the diatonic or chromatic genera, or, in other cases depending on the CIs, to either the enharmonic or chromatic. An example is the intense chromatic or soft

diatonic types, where the interval near 250 cents may be perceived as either a large whole tone or a small minor third. This type of ambiguity may be made compositionally significant in a piece employing many different tetrachords.

The middle portion of the chapter deals with various types of harmonic and melodic distance functions between tetrachords having different intervals or intervallic arrangements. Included in this section is a discussion of the statistical properties of tetrachords, including various means (geometric mean, harmonic mean, and root mean square; see chapter 4) and statistical measures of central tendency (mean deviation, standard deviation, and variance). Both tabular and graphical representations are used; the tabular is useful to produce a feeling for the actual values of the parameters.

These concepts should be helpful in organizing modulations between various tetrachords and tetrachordal scales. For example, one could cut the solid figures generated by the various means over the whole tetrachordal space by various planes at different angles to the axes. The intersections of the surfaces with the planes or the interiors of the bounded portions of the figures of intersection define sets of tetrachords. Planes parallel to the bases define tetrachordal sets with invariant values of the means, and oblique planes describe sets with limited parametric ranges. Similarly, lines (geodesics) on the surfaces of the statistical measures delineate other tetrachordal sets. These techniques are similar to that employed by Thomas Miley in his compositions *Z-View* and *Distance Music*, in which the intersections of spheres and planes defined sets of intervals (Miley 1989).

The distance functions are likewise pertinent both to manual and algorithmic composition. James Tenney has used harmonic and melodic distance functions in *Changes: Sixty-four Studies for Six Harps*, a cycle of pieces in 11-limit just intonation. Polansky's morphological metrics are among the most powerful of the distance functions. Polansky has used morphological metrics in a number of recent compositions, although he has not yet applied them to sets of tunings (Polansky, 1991, personal communication). His compositions employing morphological metrics to date are *17 Simple Melodies of the Same Length* (1987), *Distance Musics I-VI* (1987), *Duet* (1989), *Three Studies* (1989) and *Bedhaya Sadra / Bedhaya Guthrie* (1988–1991).

In the absence of any published measurements known to the author of the perceptual differences between tetrachordal genera and tetrachordal permutations, the question of which of the distance functions better models

perception is unanswerable. There may be a number of interesting research problems in the psychology of music in this area.

The chapter concludes with a discussion Rothenberg's concept of *propriety* as it applies to tetrachords and heptatonic scales derived from tetrachords. Rothenberg has used propriety and other concepts derived from his theoretical work on perception in his own compositions, i.e., *Inharmonic Figurations* (Reinhard 1987).

Historical classification

The ancient Greek theorists classified tetrachords into three genera according to the position of the third note from the bottom. This note was called *lichanos* ("indicator") in the hypaton and meson tetrachords and *paranete* in the diezeugmenon, hyperbolaion, and synemmenon tetrachords (chapter 6). The interval made by this note and the uppermost tone of the tetrachord may be called the *characteristic interval* (CI), as its width defines the genus, though actually it has no historical name. If the lichanos was a semitone from the lowest note, making the CI a major third with the 4/3, the genus was termed enharmonic. A lichanos roughly a whole tone from the 1/1 produced a minor third CI and created a chromatic genus. Finally, a lichanos a minor third from the bottom and a whole tone from the top defined a diatonic tetrachord.

The Islamic theorists (e.g., Safiyu-d-Din, 1276; see D'Erlanger 1938) modified this classification so that it comprised only two main categories translatable as "soft" and "firm." (D'Erlanger 1930; 1935) The soft genera comprised the enharmonic and chromatic, those in which the largest interval is greater than the sum of the two smaller ones, or equivalently, is greater than one half of the perfect fourth. The firm genera consisted of the diatonic, including a subclass of reduplicated forms containing repeated whole tone intervals. These main genera were further subdivided according to whether the pykna were linearly divided into approximately equal (1:1) or unequal (1:2) parts. The 1:1 divisions were termed "weak" and the 1:2 divisions, "strong."

These theorists added many new tunings to the corpus of known tetrachords and also tabulated the intervallic permutations of the genera. This led to compendious tables which may or may not have reflected actual musical practice.

Crocker's tetrachordal comparisons

Richard L. Crocker (1963, 1964, 1966) analyzed the most important of the ancient Greek tetrachords (see chapters 2 and 3) in terms of the relative magnitudes of their intervals. Crocker was interested in the relation of the older Pythagorean tuning to the innovations of Archytas and Aristoxenos. He stressed the particular emphasis placed on the position of the lichanos by Archytas who employed 28/27 as the first interval (parhypate to 1/1) in all three genera. In Pythagorean tuning, the chromatic and diatonic parhypatai are a limma (256/243, 90 cents) above hypate, while the enharmonic division is not certain. The evidence suggests a *limmatic pyknon*, but it may not have been consistently divided much prior to the time of Archytas (Winnington-Ingram 1928).

Archytas's divisions are in marked contrast to the genera of Aristoxenos, who allowed both lichanos and parhypate to vary within considerable ranges. With Archytas the parhypatai are fixed and all the distinction between the genera is carried by the lichanoi. These relations can be seen most clearly in 5-1, 5-2, and 5-3. These figures have been redrawn from those in Crocker (1966).

This type of comparison has been extended to the genera of Didymos, Eratosthenes and Ptolemy in 5-4, 5-5, and 5-6. The genera of Didymos and Eratosthenes resemble those of Aristoxenos with their pykna divided in rough equality.

Ptolemy's divisions are quite different. For Aristoxenos, Didymos, and Eratosthenes, the ratio of the intervals of the pyknon are roughly 1:1, except in the diatonic genera. Ptolemy, however, uses approximately a 2:1 relationship.

Barbera's rate of change function

C. André Barbera (1978) examined these relations in more detail. He was especially interested in the relations between the change in the position of the lichanoi compared to the change in the position of the parhypatai as one moved from the enharmonic through the chromatic to the diatonic genera. Accordingly, he defined a function over pairs of genera which compared the change in the location of the lichanoi to the change in that of the parhypatai. His function is (lichanos$_2$ – lichanos$_1$) / (parhypate$_2$ – parhypate$_1$) where the corresponding notes of two tetrachords are subscripted. This function is meaningful only when computed on a series of related genera

5-1. *Archytas's genera. These genera have a constant 28/27 as their parhypate.*

ENHARMONIC

28/27	36/35	5/4	
0	63	112	498

CHROMATIC

28/27	243/224	32/27	
0	63	204	498

DIATONIC

28/27	8/7	9/8	
0	63	294	498

5-2. *Pythagorean genera. These genera are traditionally attributed to Pythagoras, but in fact are of Babylonian origin (Duchesne-Guillemin 1963, 1969). The division of the enharmonic pyknon is not known, but several plausible tunings are listed in the Main Catalog.*

ENHARMONIC

?	?	81/64	
0	?	90	498

CHROMATIC

256/243	2187/2048	32/27	
0	90	204	498

DIATONIC

256/243	9/8	9/8	
0	90	294	498

5-3. *Aristoxenos's genera, expressed in Cleonides's parts rather than ratios. One part equals 16.667 cents.*

ENHARMONIC

0	50	100	500

3 + 3 + 24 PARTS

SOFT CHROMATIC

0	67	133	500

4 + 4 + 22 PARTS

HEMIOLIC CHROMATIC

0	75	150	500

4.5 + 4.5 + 21 PARTS

INTENSE CHROMATIC

0	100	200	500

6 + 6 + 18 PARTS

SOFT DIATONIC

0	100	250	500

6 + 9 + 15 PARTS

INTENSE DIATONIC

0	100	300	500

6 + 12 + 12 PARTS

5-4. Didymos's genera. Didymos's chromatic is probably the most consonant tuning for the 6/5 genus. His diatonic differs from Ptolemy's only in the order of the 9/8 and 10/9.

ENHARMONIC

32/31	31/30	5/4	
0	55	112	498

CHROMATIC

16/15	25/24	6/5	
0	112	183	498

DIATONIC

16/15	10/9	9/8	
0	112	294	498

5-5. Eratosthenes's genera. Eratosthenes's diatonic is the same as Ptolemy's ditone diatonic.

ENHARMONIC

40/39	39/38	19/15	
0	44	89	498

CHROMATIC

20/19	19/18	6/5	
0	89	183	498

DIATONIC

256/243	9/8	9/8	
0	90	294	498

5-6. Ptolemy's genera. Only Ptolemy's own genera are shown. Ptolemy's tonic diatonic is the same as Archytas's diatonic. His ditone diatonic is the Pythagorean diatonic.

ENHARMONIC

46/45	24/23	5/4	
0	38	113	498

SOFT CHROMATIC

28/27	15/14	6/5	
0	63	182	498

INTENSE CHROMATIC

22/21	12/11	7/6	
0	81	232	498

SOFT DIATONIC

21/20	10/9	8/7	
0	85	267	498

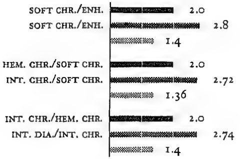
INTENSE DIATONIC

16/15	9/8	10/9	
0	112	316	498

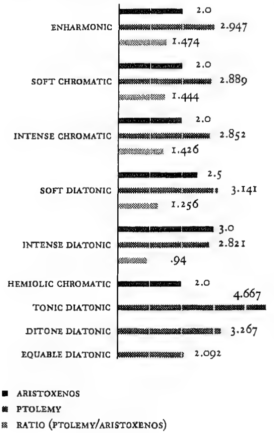
EQUABLE DIATONIC

12/11	11/10	10/9	
0	151	316	498

5-7. *Barbera's function applied to Aristoxenos's and Ptolemy's genera.*



5-8. *Ratio of lichanos to parhypate in Aristoxenos's and Ptolemy's genera.*



such as Aristoxenos's enharmonic and his chromatics or on the corresponding ones of Ptolemy. The extent to which such calculations give consistent values is a measure of the relatedness of the tetrachordal sets.

In 5-7, the results of such calculations are shown. The value for Aristoxenos's non-diatonic genera is 2.0. Ptolemy's genera yield values near 3.0, and the discrepancies are due to his use of superparticular ratios and just intonation rather than equal temperament. The proportion of the Ptolemaic to the Aristoxenian values is near 1.4.

These facts suggest that both theorists conceived their tetrachords as internally related sets, not as isolated tunings. Presumably, the increase from 2.0 to about 3 of this parameter reflects a change in musical taste in the nearly 500 years elapsed between Aristoxenos and Ptolemy.

Both ancient theorists presented additional genera not used in this computation. Some, such as Aristoxenos's hemiolic chromatic or Ptolemy's equable diatonic, had no counterpart in the other set. Ptolemy's soft diatonic appears to be only a variation or inflection of his intense (syntonic) chromatic. His remaining two diatonics, the tonic and ditonic, were of historical origin and not of his invention. The same is true of Aristoxenos's intense diatonic which seems clearly intended to represent the archaic ditone or Pythagorean diatonic.

A comparison of the corresponding members of these two authors' sets of tetrachords by a simpler function is also illuminating. If one plots the ratio of lichanos to parhypate or, equivalently, the first interval versus the sum of the first two, it is evident that Aristoxenos preferred an equal division of the pyknon and Ptolemy an unequal 1:2 relation. These preferences are shown by the data in 5-8, where the lichanos/parhypate ratio is 2.0 for Aristoxenos's tetrachords and about 3.0 for Ptolemy's non-diatonic genera.

One may wonder whether Ptolemy's tetrachords are theoretical innovations or whether they faithfully reflect the music practice of second century Alexandria. The divisions of Didymos and Eratosthenes, authors who lived between the time of Aristoxenos and Ptolemy, resemble Aristoxenos's, and there are strong reasons to assume that Aristoxenos is a trustworthy authority on the music of his period (chapter 3). The lyra and kithara scales he reports as being in use by contemporary musicians would seem to indicate that the unequally divided pyknon was a musical reality (chapter 6). Ptolemy's enharmonic does seem to be a speculative

construct as the enharmonic genus was extinct by the third century BCE (Winnington-Ingram 1932). His equable diatonic, however, resembles modern Islamic scales and certain Greek orthodox liturgical tetrachords (chapter 3).

These historical studies are important not only for what they reveal about ancient musical thought but also because they are precedents for organizing groups of tetrachords into structurally related sets. The use of constant or contrasting pyknotic/apyknotic proportions can be musically significant. Modulation of genus (μεταβολε κατα γενοσ) from diatonic to chromatic or enharmonic and back was a significant stylistic feature of ancient music according to the theorists. Several illustrations of this technique are found among the surviving fragments of Greek music (Winnington-Ingram 1936).

5-9. *Neo-Aristoxenian classification.* a + b + c = 500 *cents. This classification is based on the size of the largest or characteristic interval (CI); the equal division of the pyknon* (a+b) *is only illustrative and other divisions exist. The hyperenharmonic genera have CIs between the major third and the fourth and pyknotic intervals of commatic size. The enharmonic genera contain CIs approximating major thirds. The chromatic genera range from the soft chromatic to the soft diatonic of Aristoxenos or the intense chromatic of of Ptolemy. The diatonic are all those genera without pykna, i.e., whose largest interval is less than 250 cents.*

HYPERENHARMONIC

$c/10 < a + b \leq 3c/17$

23+23+454 *to* 37.5+37.5+425 *cents*

80/79 · 79/78 · 13/10 *to* 50/49 · 49/48 · 32/25

ENHARMONIC

$3c/17 < a + b \leq c/3$

37.5+37.5+425 *to* 62.5+62.5+375 *cents*

48/47 · 47/46 · 23/18 *to* 30/29 · 29/28 · 56/45

CHROMATIC

$c/3 < a + b \leq c$

62.5+62.5+375 *to* 125+125+250 *cents*

29/28 · 28/27 · 36/29 *to* 15/14 · 14/13 · 52/45

DIATONIC

$c < a + b \leq 2c$

125+125+250 *to* 167+167+167 *cents*

104/97 · 97/90 · 15/13 *to* 11/10 · 11/10 · 400/363

Neo-Aristoxenian classification

The large number of new tetrachordal divisions generated by the methods of chapter 4 indicates a need for new classification tools. A conveniently simple scheme is the neo-Aristoxenian classification which assumes a tempered fourth of 500 cents and categorizes tetrachords into four classes according to the sizes of their CIs. For tetrachords in just intonation, the fourth has 498.045 cents, and the boundaries between categories will be slightly adjusted. The essential feature of this scheme is the geometrical approach of chapter three.

Those new genera whose CIs fall between a major third and perfect fourth may be denoted *hyperenharmonic* after Ervin Wilson (personal communication) who first applied it to the 56/55 · 55/54 · 9/7 genus. The hyperenharmonic CIs range from roughly 450 cents down to 425 cents. The next class is the enharmonic with CIs ranging from 425 to 375 cents, a span of 50 cents. The widest division is the chromatic, from 375 cents to 250 cents as it includes CIs whose widths vary from the neutral thirds of approximately 360–350 cents (16/13, 11/9, 27/22) through the minor and subminor thirds (6/5, 7/6) to the "half-augmented seconds" (15/13, 52/45) near 250 cents. Beyond this limit, a pyknon no longer exists and the genera are diatonic.

This neo-Aristoxenian classification is summarized in 5-9. The limits of the categories are illustrated with representative tetrachords in just intonation.

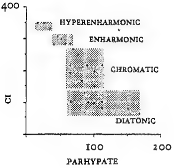
These four main classes may be further subdivided according to the proportions of the two intervals which divide the pyknon, or apyknon in the case of the diatonic genera. Because of the large number of possible divisions, it is clearer and easier to display the various subgenera graphically than to try to name them individually. Thus a number of representative tetrachords from the Main Catalog have been plotted in 5-10–12 to illustrate the most important types.

In 5-10, the first interval, as defined by the position of the note *parhypate*, has been plotted against the characteristic interval. For most of the historical tetrachords of chapters 2 and 3, this is equivalent to plotting the smallest versus the largest intervals or the first against the third. The exceptions, of course, are Archytas's enharmonic and diatonic and Didymos's chromatic.

5-11 shows the position of the third note, lichanos, graphed against the second, parhypate. This is equivalent to comparing the size of the whole pyknon (or *apyknon*) to its first interval. This particular display recalls the Greek classification by the position of the lichanoi and the differentiation into shades or chroai by the position of the parhypatai.

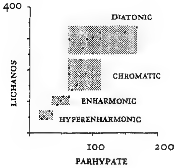
The first interval is plotted against the second in 5-12. In this graph, however, all of the permutations of this set of typical tetrachords are also plotted. This type of plot reveals the inequality of intervallic size between genera and distinguishes between permutations when the tetrachords are not in the standard Greek ascending order of smallest, medium, and large.

5-10.



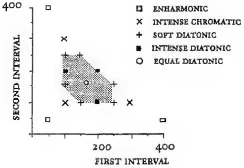
5-10. *Plot of characteristic intervals versus parhypatai. The four notes of the illustrative meson tetrachord in ascending order of pitch are hypate, parhypate, lichanos, and mese. The CI is the interval between lichanos and mese.*

5-11.



5-11. *Plot of lichanoi versus parhypatai.*

5-12.



5-12. *First interval plotted against second intervals of major tetrachordal genera. The tetrachords plotted here are 50 + 50 + 400, 100 + 100 + 300, 100 + 150 + 250, 100 + 200 + 200, and 166.67 + 166.67 + 166.67 cents in all of their intervallic permutations. The permutations of the soft diatonic genus delineate the region of Rothenberg-proper diatonic scales.*

5-13. *Intervallic inequality functions on just and tempered tetrachords.*

RATIOS	CI/MIN	CI/MID	MID/MIN
HYPERENHARMONIC			
56/55 · 55/54 · 9/7	13.95	13.70	1.018
ENHARMONIC			
28/27 · 36/35 · 5/4	7.921	66.136	1.291
32/31 · 31/30 · 5/4	7.028	6.805	1.033
46/45 · 24/23 · 5/4	10.15	5.243	1.936
CHROMATIC			
20/19 · 19/18 · 6/5	3.554	3.372	1.054
28/27 · 15/14 · 6/5	5.013	2.642	1.897
26/25 · 25/24 · 16/13	5.294	5.086	1.041
39/38 · 19/18 · 16/13	7.994	3.840	2.081
24/23 · 23/22 · 11/9	4.715	4.514	1.044
34/33 · 18/17 · 11/9	6.722	3.511	1.915
16/15 · 15/14 · 7/6	2.389	2.234	1.069
22/21 · 12/11 · 7/6	3.314	1.772	1.870
DIATONIC			
14/13 · 13/12 · 8/7	1.802	1.668	1.080
21/20 · 10/9 · 8/7	2.737	1.267	2.159
28/27 · 9/8 · 8/7	3.672	1.133	3.239
16/15 · 10/9 · 9/8	1.825	1.118	1.633
256/243 · 9/8 · 9/8	2.260	1.000	2.260
12/11 · 11/10 · 10/9	1.211	1.105	1.095
TEMPERED TETRACHORDS			
50 + 50 + 400	8.00	8.00	1.00
66.67 + 133.33 + 300	4.50	2.25	2.00
100 + 100 + 300	3.00	3.00	1.00
100 + 150 + 250	2.50	1.67	1.50
100 + 200 + 200	2.00	1.00	2.00
166.67 + 166.67 + 166.67	1.00	1.00	1.00

Intervallic inequality functions

More quantitative measures of intervallic inequality are seen in 5-13. The first measure is the ratio of the logarithms of the largest interval to that of the smallest. In practice, cents or logarithms to any base may be used. This ratio measures the extremes of intervallic inequality. The second measure is the ratio of the largest to the middle-sized interval. For tetrachords with reduplicated intervals, i.e., 256/243 · 9/8 · 9/8 or 16/15 · 16/15 · 75/64, the middle-sized interval is the reduplicated one, and this function is equal to one of the other two functions. The third measure is the ratio of the middle-sized interval to the smallest. This function often indicates the relative sizes of the two intervals of the pyknon and distinguishes subgenera with the same CI.

These functions measure the degree of inequality of the three intervals and may be defined for tetrachords in equal temperament as well as in just intonation. All of these functions are invariant under permutation of intervallic order.

Harmonic complexity functions

In addition to being classified by intervallic size, tetrachords may also be characterized by their harmonic properties. Although harmony in the sense of chords and chordal sequences is discussed in detail in chapter 7, it is appropriate in this chapter to discuss the harmonic properties of the tetrachordal intervals in terms of the prime numbers which define them.

The simplest harmonic function which may be defined on a tetrachord or over a set of tetrachords is the largest prime function. The value of this function is that of the largest prime number greater than 2 in the numerators or denominators of three ratios defining the tetrachord. The tetrachord (or any other set of intervals) is said to have an *n-limit* or be an *n*-limit construct when *n* is the largest prime number in the defining ratio(s), irrespective of its exponent and the exponent's sign.

One limitation of the *n*-limit function is that it uses only a small part of the information in the tetrachordal intervals. As a result, numerous genera with different melodic properties have the same *n*-limit. However, this one-dimensional descriptor is often used by composers of music in just intonation (David Doty, personal communication). For example, the following diverse set of tetrachords all contain 5 as their largest prime number: 25/24 · 128/125 · 5/4, 256/243 · 81/80 · 5/4, 16/15 · 25/24 · 6/5, 256/243 ·

5-14. *Harmonic complexity and simplicity functions on tetrachords in just intonation. (1) CI complexity: the sum of the prime factors of the largest interval. (2) Pyknotic complexity: the joint complexity of the two intervals of the pyknon. (3) Average complexity: the arithmetic mean of the CI and pyknotic complexities. (4) Total complexity: the joint complexity of the entire tetrachord. (5) Harmonic simplicity: 1 over the sum of the prime factors greater than 2 of the ratio defining the CI. It has been normalized by dividing by 0.2, as the maximum value of the unscaled function is 0.2, corresponding to 5/4 whose Wilson's complexity is 5.*

RATIOS	1	2	3	4	5
HYPERENHARMONIC					
56/55 · 55/54 · 9/7	13	32	22.5	32	.3846
ENHARMONIC					
28/27 · 36/35 · 5/4	5	21	13	21	1.000
32/31 · 31/30 · 5/4	5	39	22	39	1.000
46/45 · 24/23 · 5/4	5	34	19.5	34	1.000
CHROMATIC					
20/19 · 19/18 · 6/5	8	30	19	30	.6250
28/27 · 15/14 · 6/5	8	21	14.5	21	.6250
26/25 · 25/24 · 16/13	13	26	19.5	26	.3846
39/38 · 19/18 · 16/13	13	38	25.5	38	.3846
24/23 · 23/22 · 11/9	17	37	27	40	.2941
34/33 · 18/17 · 11/9	17	34	25.5	34	.2941
16/15 · 15/14 · 7/6	10	15	12.5	15	.5000
22/21 · 12/11 · 7/6	10	21	15.5	21	.5000
DIATONIC					
14/13 · 13/12 · 8/7	7	23	15	23	.7143
21/20 · 10/9 · 8/7	7	18	12.5	18	.7143
28/27 · 9/8 · 8/7	7	16	11.5	16	.7143
16/15 · 10/9 · 9/8	6	11	8.5	11	.8333
256/243 · 9/8 · 9/8	6	15	10.5	15	.8333
12/11 · 11/10 · 10/9	11	19	15	22	.4545

135/128 · 6/5, 16/15 · 75/64 · 16/15, 10/9 · 10/9 · 27/25, and 16/15 · 9/8 · 10/9. Similarly, all the Pythagorean tunings in the Catalog are at the 3-limit.

The second limitation of the largest prime number function when applied to the whole tetrachord is that it does does not distinguish between intervals which may be of differing harmonic importance to the composer. Primary distinctions between genera are determined by the sizes of their characteristic intervals. Genera with similarly sized CIs may have quite different musical effects due to the different degrees of consonance of these intervals. Similar effects are seen with the pyknotic intervals as well, particularly those due to the first interval which combines with mese or the added note, hyperhypate, to form an interval characteristic of the oldest Greek styles (Winnington-Ingram 1936 and chapter 6). In these cases, the largest prime function must be applied to the individual intervals and not just to the tetrachord as a whole.

For these reasons, other indices of harmonic complexity have been developed which utilize more of the information latent in the tetrachordal intervals. These indices have been computed on a representative set of tetrachords and their component intervals. The first of the indices is Wilson's *complexity* function which for single intervals may be defined as the sum of their prime factors (greater than 2) times the absolute values of their exponents. For example, the complexities of 3/2 and 4/3 are both 3 and those of 6/5 and 5/3 are both 8 (3 + 5). Similarly, the intervals 9/7 and 14/9 both have complexities of 13 (3 + 3 + 7). The complexities of the CIs of some important genera are tabulated in 5-14.

Wilson's complexity function may also be applied to sets of intervals by finding the modified least common multiple of the prime factors (with all the exponents made positive). The pyknon of Archytas's enharmonic consists of the intervals 28/27 and 36/35. The first ratio may be expressed as $7 + 3^3$ and the second as $3^2 + 5 + 7$. The modified least common multiple of this set is $3^3 \cdot 5 \cdot 7$ and the Wilson's complexity is 21 (3 + 3 + 3 + 5 + 7). The average complexity, which is the arithmetic mean of the complexities of the CI and the pyknon, and the total complexity, which is the joint complexity of all three intervals, are also shown in 5-14. In most cases the latter index equals the pyknotic complexity.

An alternative index which may be more convenient in some cases is the harmonic *simplicity*, which is the reciprocal of the complexity. This function

5-15. Euclidean distances between genera in just intonation. The upper set of numbers is the distance calculated on the largest versus the smallest intervals of the tetrachords. The lower set is computed from the first and second intervals. The Euclidean distance is the square root of the sum of the squares of the differences between corresponding intervals. Values are in cents.

5-16. Euclidean distances between tempered genera. The 1:2 chromatic is the "strong" form corresponding to the intense chromatic of Aristoxenos. The equal diatonic is 166.67 + 166.67 + 166.67 cents.

may be normalized, as it is in 5-14, by dividing its values by 5, which is the maximum simplicity of a CI or tetrachord (because 5/4 is the simplest interval smaller than 4/3).

Euclidean distances between tetrachords

The methods described in chapter 4 and in the compilations of the historical authors provide many tetrachords with diverse melodic characteristics. To bring some order to these resources, some measure of the perceptual distance between different genera or between different permutations of the same genus is desirable. While a useful measure of the distance between genera may be obtained from the differences between the characteristic intervals, this measure does not distinguish between the subgenera (i.e., the 1:1 and 1:2 divisions of the pyknon). A more precise measure is afforded by the Euclidean distances between genera on a plot of the CI versus the

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	72.09 70.67	73.99 63.43	123.59 103.37	192.96 162.62	227.94 145.59
28/27 · 15/14 · 6/5		7.71 10.91	51.84 35.81	121.91 97.54	159.50 98.81
25/24 · 16/15 · 6/5			49.76 40.14	119.04 100.91	155.39 96.09
22/21 · 12/11 · 7/6				70.26 61.73	109.77 71.56
16/15 · 9/8 · 10/9					44.45 55.02

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	101.36 84.89	111.80 70.71	158.11 111.80	206.16 158.11	260.87 164.99
1:2 CHROMATIC (67 + 133 + 300)		33.33 47.14	60.09 37.27	105.41 74.54	166.67 105.41
INTENSE CHROMATIC (100 + 100 + 300)			50.0 50.0	100.0 100.0	149.07 94.28
SOFT DIATONIC (100 + 150 + 250)				50.0 50.0	106.72 68.72
INTENSE DIATONIC (100 + 200 + 200)					74.54 74.54

5-17. Euclidean distances between permutations of Archytas's enharmonic genus. The function tabulated is the distance calculated on the plot of the first by the second interval of the tetrachord. The other distance function, computed from the graph of the greatest versus the least interval, is always zero between permutations of the same genus.

5-18. Euclidean distances between permutations of tempered genera.

smallest interval or of the first versus the second interval.

The distances are calculated according to the Pythagorean relation: the distance is defined as the square root of the sum of the squares of the differences of the coordinates. The Euclidean distance is $\sqrt{[(CI_2 - CI_1)^2 + (\text{parhypate}_2 - \text{paryhypate}_1)^2]}$ in the first case and $\sqrt{[(\text{first interval}_2 - \text{first interval}_1)^2 + (\text{second interval}_2 - \text{second interval}_1)^2]}$ in the second. It is convenient to convert the ratios into cents for these calculations. The distances between some representative tetrachords in just intonation are tabulated in 5-15 and some in equal temperament with similar melodic contours in 5-16.

One may also use the second Euclidean distance function to distinguish between permutations of tetrachords as shown in 5-17 and 5-18.

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	337.54	337.84	20.07	323.66	323.35
28/27 · 5/4 · 36/35		14.19	323.66	457.29	467.43
36/35 · 5/4 · 28/27			323.55	467.43	155.39
36/35 · 28/27 · 5/4				337.54	337.84
5/4 · 28/27 · 36/35					14.19

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	350.0	350.0
50 + 400 + 50		494.97

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	200.0	200.0
100 + 300 + 100		282.84

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	141.42	100.0
200 + 100 + 200		100.0

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	100.0	70.71	111.81	158.11	150.0
100 + 250 + 150		158.11	50.0	212.13	180.28
150 + 100 + 250			150.0	100.0	111.80
150 + 250 + 100				180.28	141.42
250 + 100 + 150					50.0

Minkowskian distances between tetrachords

The closely related *Minkowski metric* or *city block* distance function is shown in 5-19 and 5-20 for the same sets of tetrachords. The two functions shown here are defined as the sum of the absolute values of the differences between corresponding intervals. For the upper set of numbers, the function is $(|\mathrm{CI}_2 - \mathrm{CI}_1| + |\text{parhypate}_2 - \text{paryhypate}_1|)$ and for the lower set, $(|\text{first interval}_2 - \text{first interval}_1| + |\text{second interval}_2 - \text{second interval}_1|)$. These computations have also been done in cents throughout for ease of comparison.

The distances between permutations may also be compared by means of the second distance function (5-21 and 5-22).

5-19. *Minkowski or "city block" distances between genera in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	84.86 70.67	92.57 70.67	151.21 119.44	245.36 203.91	305.78 203.91
28/27 · 15/14 · 6/5		7.71 15.42	66.35 48.77	160.50 133.24	220.91 133.24
25/24 · 16/15 · 6/5			58.64 48.77	152.79 133.24	213.20 133.24
22/21 · 12/11 · 7/6				94.16 84.47	109.77 84.47
16/15 · 9/8 · 10/9					77.81 60.41

5-20. *Minkowski or "city block" distances between tempered genera.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	116.67 100.0	150.0 100.0	200.0 150.0	250.0 200.0	350.0 233.33
1:2 CHROMATIC (67 + 133 + 300)		33.33 66.67	83.33 50.0	133.33 100.0	233.33 200.0
INTENSE CHROMATIC (100 + 100 + 300)			50.0 50.0	100.0 100.0	200.0 133.33
SOFT DIATONIC (100 + 150 + 250)				50.0 50.0	150. 83.33
INTENSE DIATONIC (100 + 200 + 200)					100.0 100.0

5-21. *Minkowski or "city block" distances between permutations of Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	337.54	351.73	28.38	337.54	323.35
28/27 · 15/14 · 6/5		14.19	337.54	646.71	660.90
25/24 · 16/15 · 6/5			323.35	660.90	675.09
22/21 · 12/11 · 7/6				337.54	351.73
16/15 · 9/8 · 10/9					14.19

5-22. *Minkowski or "city block" distances between permutations of tempered genera.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	350.0	350.0
100 + 250 + 150		700.0

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	200.0	200.0
100 + 300 + 100		400.0

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	200.0	100.0
200 + 100 + 200		100.0

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	100.0	100.0	150.0	200.0	150.0
100 + 250 + 150		200.0	50.0	300.0	250.0
150 + 100 + 250			150.0	100.0	150.0
150 + 250 + 100				250.0	200.0
250 + 100 + 150					50.0

5-23. *Tenney pitch and harmonic distance funcions on the intervals of tetrachords in just intonation.*

	C.I.'S	MID	SMALL
56/55 · 55/54 · 9/7	0.109 1.799	.0080 3.473	.0078 3.489
28/27 · 36/35 · 5/4	.0969 1.301	.0158 2.879	.0122 3.100
32/31 · 31/30 · 5/4	.0969 1.301	.0142 2.968	.0138 2.997
46/45 · 24/23 · 5/4	.0969 1.301	.0184 2.742	.0096 3.156
20/19 · 19/18 · 6/5	.0792 1.477	.0235 2.534	.0223 2.580
28/27 · 15/14 · 6/5	.0792 1.477	.0300 2.322	.0158 2.878
26/25 · 25/24 · 16/13	.0902 2.318	.0177 2.778	.0170 2.813
39/38 · 19/18 · 16/13	.0902 2.318	.0235 2.534	.0113 3.171
24/23 · 23/22 · 11/9	.0872 1.996	.0193 2.704	.0185 2.742
34/33 · 18/17 · 11/9	.0872 1.996	.0248 2.486	.0130 3.050
16/15 · 15/14 · 7/6	.0669 1.623	.0300 2.322	.0280 2.380
22/21 · 12/11 · 7/6	.0669 1.623	.0378 2.121	.0202 2.664
14/13 · 13/12 · 8/7	.0580 1.748	.0348 2.193	.0322 2.260
21/20 · 10/9 · 8/7	.0580 1.748	.0458 1.954	.0212 2.623
28/27 · 9/8 · 8/7	.0580 1.748	.0512 1.857	1.580 2.879
16/15 · 10/9 · 9/8	.0511 1.857	.0458 1.954	.0280 2.380
256/243 · 9/8 · 9/8	.0511 1.857	.0511 1.857	.0226 4.794
12/11 · 11/10 · 10/9	.0458 1.954	.0414 2.041	.0378 2.121

Tenney's pitch and harmonic distance functions

The composer James Tenney has developed two functions to compare intervals (Tenney 1984), and has used these functions in composition, particularly in *Changes: Sixty-four Studies for Six Harps*. The first function is the *pitch-distance* function defined as the base-2 logarithm of *a/b* where *a* and *b* are the numerator and denominator respectively of the interval in an extended just intonation. This function is equivalent to Ellis's cents which are 1200 times the base-2 logarithm. The second function is his *harmonic distance*, defined as the logarithm of $a \cdot b$. This distance function is a special use of the Minkowski metric in a tonal space where the units along each of the axes are the logarithms of prime numbers. Thus the pitch distance of the interval 9/7 is log (9/7) and the harmonic distance is 2 · log (3) + log (7).

These functions may be used to characterize tetrachords by computing distances for each of the three intervals. This has been done for the set of representative tetrachords in 5-23. The upper set of numbers is the pitch distances; the lower, the harmonic distances. Alternatively, one could also apply it to the notes of the tetrachord after fixing the tonic and calculating the notes from the successive intervals.

By a slight extension of the definition, the pitch distance function may also be applied to tempered intervals. The pitch distance is the tempered interval expressed as a logarithm. For intervals expressed in cents, the formula is pitch distance = cents / 1200 log (2); other logarithmic measures could be used. This function will be most interesting for intervals which are close approximations to those in just intonation. The harmonic distance function is not well defined for tempered intervals unless they closely approximate just intervals.

The Tenney functions also may be used to measure the distance between tetrachords. The pitch distance between the CIs of two genera is the logarithm of the quotient of their ratios; i.e., the pitch distance between 5/4, the CI of the enharmonic, and 6/5, the CI of the intense chromatic, is the logarithm of 25/24. The harmonic distance is the logarithm of 3/2, the product of 5/4 and 6/5.

The pitch distance and harmonic distance functions on the CIs distinguish genera quite well, though obviously not permutations of the genera. The Tenney distance functions between representative set of tetrachords in just intonation are shown in 5-24. One could also apply the

Tenney distance functions on the pyknotic intervals to distinguish subgenera with the same CI.

The distances between tetrachords in equal temperament may also be measured by the Tenney functions. The pitch distance of the CIs is simply the difference in cents or tempered degrees. The harmonic distance is the sum of the CIs. Data on representative tempered tetrachords are shown in 5-25.

5-24. *Tenney pitch and harmonic distances between genera in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	.0177 .1761	.0177 .1761	.0270 .1638	.0458 .1481	.0512 .1427
28/27 · 15/14 · 6/5		0.0 .1584	.0122 .1461	.0280 .1303	.0334 .1249
25/24 · 16/15 · 6/5			.0122 .1461	.0280 .1303	.0334 .1249
22/21 · 12/11 · 7/6				.0158 .1181	.0212 .1121
16/15 · 9/8 · 10/9					.0054 .0969

5-25. *Tenney pitch and harmonic distances between tempered genera.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC 50 + 50 + 400	100.0 700.0	100.0 700.0	150.0 650.0	200.0 600.0	233.33 566.67
1:2 CHROMATIC 67 + 133 + 300		0.0 600.0	50.0 550.0	100.0 500.0	133.33 466.67
INTENSE CHROMATIC 100 + 100 + 300			50.0 550.0	100.0 500.0	133.33 466.67
SOFT DIATONIC 100 + 150 + 250				50.0 450.0	83.33 416.67
INTENSE DIATONIC 100 + 200 + 200					33.33 366.67

5-27. *Barlow's specific harmonicity function on tetrachords and tetrachordal scales. The specific harmonicity function is the square of the number of tones in the scale divided by sum of the reciprocals of the harmonicities of the combinatorial intervals (Barlow 1987) without regard to sign. For the tetrachord, the number of tones is 4,* $n^2 = 16$, *and there are six combinatorial intervals (see 5-28). The specific harmonicity of the Dorian mode is defined as above save that* $n = 8$ *(including the octave),* $n^2 = 64$, *and there are 28 intervals* $(n \cdot (n-1)/2)$.

	RATIOS	TETRACHORD	DORIAN
1.	56/55 · 55/54 · 9/7	.1063	.0973
2.	28/27 · 36/35 · 5/4	.1859	.1633
3.	32/31 · 31/30 · 5/4	.0724	.0660
4.	46/45 · 24/23 · 5/4	.0885	.0815
5.	20/19 · 19/18 · 6/5	.1042	.0946
6.	28/27 · 15/14 · 6/5	.1911	.1721
7.	26/25 · 25/24 · 16/13	.1062	.0998
8.	39/38 · 19/18 · 16/13	.0719	.0677
9.	24/23 · 23/22 · 11/9	.0767	.0698
10.	34/33 · 18/17 · 11/9	.0848	.0807
11.	16/15 · 15/14 · 7/6	.2170	.1879
12.	22/21 · 12/11 · 7/6	.1375	.1274
13.	14/13 · 13/12 · 8/7	.1247	.1143
14.	21/20 · 10/9 · 8/7	.1739	.1627
15.	28/27 · 9/8 · 8/7	.2101	.1885
16.	16/15 · 10/9 · 9/8	.2658	.2363
17.	256/243 · 9/8 · 9/8	.2212	.2025
18.	12/11 · 11/10 · 10/9	.1609	.1437
19.	11/10 · 11/10 · 400/363	.0829	.0797
20.	16/15 · 25/24 · 6/5	.2374	.2133

factor of $2 \cdot \xi(hcf)$, where *hcf* is the highest common factor, must be subtracted from the denominator of the formula.

Barlow's harmonicity function is applied to set of tetrachords in just intonation in 5-26. The harmonicities of the three intervals are computed separately. The harmonicity of 4/3 is the constant –0.2143. The harmonicities of the pykna are also included to complete the characterization of the tetrachords.

In the case of the general tetrachord $a \cdot b \cdot c$, where $c = 4/3ab$, there are four ratios, 1/1, a, $a \cdot b$, and 4/3. The $n \cdot (n-1)/2 = 6$ combinatorial intervals are a, ab, 4/3, b, $4/3a$, and $4/3ab$. For example, Archytas's enharmonic, 28/27 · 36/35 · 5/4, yields the tones 1/1, 28/27, 16/15, and 4/3. The combinatorial intervals are 28/27, 16/15, 4/3, 36/35, 9/7, and 5/4 the six non-redundant differences between the four tones of the tetrachord. The definition of these intervals for equally tempered tetrachords is shown as the Polansky set in 5-48. In just intonation, the sums and differences become products and quotients and the zero and 500 cents are replaced by 1/1 and 4/3 respectively.

For scales and other sets of ratios, Barlow defined a third function, termed *specific harmonicity*. The specific harmonicity of a set of ratios is the square of the number of tones divided by the sum of the absolute values of the reciprocals of the harmonicities of the combinatorial intervals (Barlow 1987). For the tetrachord, $n = 4$ and $n^2 = 16$. The specific harmonicities are presented in 5-27–29 for various sets of tetrachords.

Similarly, the specific harmonicities of scales generated from tetrachords may be computed. In the case of heptatonic scales, there are eight tones including the octave (2/1) and 28 combinatorial relations, which are defined analogously to the six of the tetrachord. The specific harmonicities of the same set of tetrachords as in 5-26 are given in 5-27. The specific harmonicities of both the tetrachords and a representative heptatonic scale are included in this table.

The Dorian mode was selected for simplicity, but other scales could have been used as well (see chapter 6 for a detailed discussion of scale construction from tetrachords). It is the scale composed of an ascending tetrachord, a 9/8 tone, and an identical tetrachord which completes the octave. Abstractly, the tones are 1/1 a ab 4/3 3/2 $3a/2$ $3ab/2$ 2/1, where $a \cdot b \cdot 4/3ab$ is the generalized tetrachord in just intonation. The set of combinatorial intervals is a, ab, 4/3, 3/2, $3a/2$, $3ab/2$, 2/1, b, $4/3a$, $3/2a$, 3/2, $3b/2$, $2/a$, $4/3ab$, $3/2ab$, $3/2b$,

$3/2$, $2/ab$, $9/8$, $9a/8$, $9ab/8$, $3/2$, a, ab, $4/3$, b, $4/3a$, $4/3ab$. The repeated intervals are a consequence of the modular structure of tetrachordal scales.

As can be seen from 5-27, the specific harmonicity function distinguishes different tetrachords and their derived scales quite well. 5-28 shows the results of an attempt to use this function to distinguish permutations of tetrachords from each other. Although the specific harmonicity function does not differentiate between intervallic retrogrades ($a \cdot b \cdot c$ versus $c \cdot b \cdot a$) of single tetrachords, it is quite effective when applied to the corresponding heptatonic scales.

Finally, since the specific harmonicity function is basically a theoretical measure of consonance, it would be interesting to use it to determine the most consonant tunings or shades (chroai) of the various genera. Accordingly, a number of tetrachords whose intervals had relatively "digestible" prime factors were examined. The results are tabulated in 5-29. It is clear that while the diatonic genera are generally more consonant than chromatic and they in turn are more harmonious than the enharmonic, there is considerable overlap between genera and permutations.

In particular, the most consonant chromatic genera are more consonant than many of the diatonic tunings.

5-28. *Barlow's specific harmonicity function on the permutations of Ptolemy's intense diatonic genus.*

	RATIOS	TETRACHORD	DORIAN
1.	16/15 · 9/8 · 10/9	.2794	.2567
2.	16/15 · 10/9 · 9/8	.2658	.2363
3.	9/8 · 10/9 · 16/15	.2658	.2535
4.	9/8 · 16/15 · 10/9	.2586	.2407
5.	10/9 · 16/15 · 9/8	.2586	.2398
6.	10/9 · 9/8 · 16/15	.2794	.2486

5-29. *The most consonant genera according to Barlow's specific harmonicity function.*

	RATIOS	TETRACHORD	DORIAN
	ENHARMONIC		
1A.	256/243 · 81/80 · 5/4	.1878	.1669
1B.	5/4 · 81/80 · 256/243	.1878	.1715
2A.	28/27 · 36/35 · 5/4	.1859	.1633
2B.	5/4 · 36/35 · 28/27	.1859	.1667
3A.	25/24 · 128/125 · 5/4	.1806	.1550
3B.	5/4 · 128/125 · 25/24	.1806	.1556
	CHROMATIC		
1A.	16/15 · 25/24 · 6/5	.2374	.2133
1B.	6/5 · 25/24 · 16/15	.2374	.2145
2.	16/15 · 75/64 · 16/15	.2317	.2008
3A.	10/9 · 81/80 · 32/27	.2290	.2046
3B.	32/27 · 81/80 · 10/9	.2290	.2035
4A.	25/24 · 27/25 · 32/27	.1926	.1745
6A.	9/8 · 64/63 · 7/6	.2137	.1937
6B.	7/6 · 64/63 · 9/8	.2137	.1903
7A.	10/9 · 36/35 · 7/6	.2032	.1783
7B.	7/6 · 36/35 · 10/9	.2032	.1797
	DIATONIC		
1A.	9/8 · 28/27 · 8/7	.2176	.2027
1B.	8/7 · 28/27 · 9/8	.2176	.1914
2A.	10/9 · 21/20 · 8/7	.2104	.1888
2B.	8/7 · 21/20 · 10/9	.2104	.1856
3A.	16/15 · 9/8 · 10/9	.2794	.2567
3B.	10/9 · 9/8 · 16/15	.2794	.2486
4A.	256/243 · 9/8 · 9/8	.2212	.2025
4B.	9/8 · 9/8 · 256/243	.2212	.2105
5.	10/9 · 27/25 · 10/9	.2251	.1993

Euler's *gradus suavitatis* function

A function somewhat similar to Wilson's, Tenney's, and Barlow's functions is Euler's *gradus suavitatis* (GS) or degree of harmoniousness, consonance, or pleasantness (Euler 1739 [1960]; Helmholtz [1877] 1954). Like the other functions, the GS is defined on the prime factors of ratios, scales, or chords.

Unlike Barlow's functions, the GS is very easy to compute. The GS of a prime number or of the ratio of a prime number relative to 1 is the prime number itself, i.e., the GS of 3/1 is 3. The GS of a composite number is the sum of the GSs of the prime factors minus one less than the number of factors. The GS of a ratio is found by first converting it to a section of the harmonic series and then computing the least common multiple of the terms. The GS of the least common multiple is the GS of the ratio.

Sets of ratios such as chords and scales may be converted to sections of the harmonic series by multiplying each element by the lowest common denominator. For example, the harmonic series form of the major triad

5-30. *Euler's gradus suavitatis function on tetrachords in just intonation. (1) is a hyperenharmonic genus, (2)–(4) are enharmonic, (5)–(12) and (20) are chromatic, and (13)–(19) are diatonic. The tetrachords are in their standard form with the small intervals at the base and the largest interval at the top. See 5-32 and 5-33 for other permutations of the tetrachord.*

	RATIOS	INTERVAL A	INTERVAL B	CI	PYKNON
1.	56/55 · 55/54 · 9/7	24	22	11	15 (28/27)
2.	28/27 · 36/35 · 5/4	15	17	7	11 (16/15)
3.	32/31 · 31/30 · 5/4	36	38	7	11 (16/15)
4.	46/45 · 24/23 · 5/4	32	28	7	11 (16/15)
5.	20/19 · 19/18 · 6/5	25	24	8	10 (10/9)
6.	28/27 · 15/14 · 6/5	15	14	8	10 (10/9)
7.	26/25 · 25/24 · 16/13	22	14	17	17 (13/12)
8.	39/38 · 19/18 · 16/13	34	24	17	17 (13/12)
9.	24/23 · 23/22 · 11/9	28	34	15	15 (12/11)
10.	34/33 · 18/17 · 11/9	30	22	15	15 (12/1)
11.	16/15 · 15/14 · 7/6	11	14	10	10 (8/7)
12.	22/21 · 12/11 · 7/6	20	15	10	10 (8/7)
13.	14/13 · 13/12 · 8/7	20	17	10	10 (7/6)
14.	21/20 · 10/9 · 8/7	15	10	10	10 (7/6)
15.	28/27 · 9/8 · 8/7	15	8	10	10 (7/6)
16.	16/15 · 10/9 · 9/8	11	10	8	12 (32/27)
17.	256/243 · 9/8 · 9/8	19	8	8	12 (32/27)
18.	12/11 · 11/10 · 10/9	15	16	10	8 (6/5)
19.	11/10 · 11/10 · 400/363	16	16	35	31 (121/100)
20.	16/15 · 25/24 · 6/5	11	14	8	10 (10/9)

$1/1$ $5/4$ $3/2$ is $4{:}5{:}6$. The least common multiple of this series is 60 and the GS of the major scale thus is 9.

The GSs of the component intervals of the usual set of tetrachords are shown in 5-30. The GS of $1/1$ is 1 and that of $4/3$ is 5. In 5-31, the GSs of both the tetrachords and the Dorian mode generated from each tetrachord are tabulated. The GSs of the Dorian mode are 3 more than the GSs of the corresponding tetrachords, reflecting the structure of the mode which has the identical series of intervals repeated at the perfect fifth.

The GS seems not to be particularly useful for distinguishing permutations of tetrachords, as evidenced by 5-32. It is noteworthy that the most harmonious arrangements of Ptolemy's intense diatonic are those which generate the major and natural minor modes (see the section on tritriadic scales in chapter 7).

As with Barlow's functions, the GS ranks the enharmonic the least harmonious of the major genera, though the most consonant tunings and arrangement overlap with those of the chromatic (5-33). Similarly, the most harmonious chromatic tunings approach those of the diatonic.

Interestingly, however, the most harmonious enharmonic tuning is $28/27 \cdot 5/4 \cdot 36/35$ and its retrograde which have the largest interval medially. The same is true for the chromatic $16/15 \cdot 6/5 \cdot 25/24$. Of the diatonic forms, the two arrangements of Ptolemy's intense diatonic with the $9/8$ medial are the most consonant.

Although the GS is an interesting and potentially useful function, it does have one weakness. Because the ratios defining small deviations from ideally consonant intervals contain either large primes or large composites, the GS of slightly mistuned consonances can become arbitrarily large. Thus the GS would predict slightly mistuned consonances to be extremely dissonant, a prediction not consistent with observation.

5-31. *Euler's gradus suavitatis function on tetrachords and tetrachordal scales. (1) is a hyperenharmonic genus, (2)–(4) are enharmonic, (5)–(12) and (20) are chromatic, and (13)–(19) are diatonic. The harmonic series representation of the Dorian mode of $16/15 \cdot 9/8 \cdot 10/9$ is $30{:}32{:}36{:}40{:}45{:}48{:}54{:}60$. Its least common multiple is 4320 and its GS is 16.*

	RATIOS	TETRACHORD	DORIAN
1.	$56/55 \cdot 55/54 \cdot 9/7$	30	33
2.	$28/27 \cdot 36/35 \cdot 5/4$	21	24
3.	$32/31 \cdot 31/30 \cdot 5/4$	42	45
4.	$46/45 \cdot 24/23 \cdot 5/4$	35	38
5.	$20/19 \cdot 19/18 \cdot 6/5$	29	32
6.	$28/27 \cdot 15/14 \cdot 6/5$	19	22
7.	$26/25 \cdot 25/24 \cdot 16/13$	27	30
8.	$39/38 \cdot 19/18 \cdot 16/13$	39	42
9.	$24/23 \cdot 23/22 \cdot 11/9$	40	43
10.	$34/33 \cdot 18/17 \cdot 11/9$	33	36
11.	$16/15 \cdot 15/14 \cdot 7/6$	17	20
12.	$22/21 \cdot 12/11 \cdot 7/6$	22	25
13.	$14/13 \cdot 13/12 \cdot 8/7$	24	27
14.	$21/20 \cdot 10/9 \cdot 8/7$	19	23
15.	$28/27 \cdot 9/8 \cdot 8/7$	16	19
16.	$16/15 \cdot 10/9 \cdot 9/8$	16	19
17.	$256/243 \cdot 9/8 \cdot 9/8$	19	22
18.	$12/11 \cdot 11/10 \cdot 10/9$	21	24
19.	$11/10 \cdot 11/10 \cdot 400/363$	35	38
20.	$16/15 \cdot 25/24 \cdot 6/5$	17	20

5-32. *Euler's gradus suavitatis function on the permutations of Ptolemy's intense diatonic genus. (1) is the prime form. (2) is the order given by Didymos.*

	RATIOS	TETRACHORD	DORIAN
1.	$16/15 \cdot 9/8 \cdot 10/9$	13	16
2.	$16/15 \cdot 10/9 \cdot 9/8$	16	19
3.	$9/8 \cdot 10/9 \cdot 16/15$	16	19
4.	$9/8 \cdot 16/15 \cdot 10/9$	16	19
5.	$10/9 \cdot 16/15 \cdot 9/8$	16	19
6.	$10/9 \cdot 9/8 \cdot 16/15$	13	16

	RATIOS	TETRACHORD	DORIAN
	ENHARMONIC		
1A.	256/243 · 81/80 · 5/4	23	26
2A.	28/27 · 36/35 · 5/4	21	24
2B.	28/27 · 5/4 · 36/35	19	22
2C.	36/35 · 28/27 · 5/4	21	24
3A.	25/24 · 128/125 · 5/4	22	25
	CHROMATIC		
1A.	16/15 · 25/24 · 6/5	17	20
1B.	25/24 · 16/15 · 6/5	18	21
1C.	16/15 · 6/5 · 25/24	16	19
2.	16/15 · 75/64 · 16/15	17	20
3A.	10/9 · 81/80 · 32/27	18	21
3B.	32/27 · 81/80 · 10/9	18	21
4A.	25/24 · 27/25 · 32/27	20	23
4B.	32/27 · 27/25 · 25/24	20	23
5A.	16/15 · 15/14 · 7/6	17	20
5B.	16/15 · 7/6 · 15/14	19	22
6A.	9/8 · 64/63 · 7/6	19	22
6B.	64/63 · 9/8 · 7/6	17	20
7A.	10/9 · 36/35 · 7/6	18	21
7B.	10/9 · 7/6 · 36/35	19	22
7C.	36/35 · 10/9 · 7/6	20	23
	DIATONIC		
1A.	9/8 · 28/27 · 8/7	18	21
1B.	8/7 · 9/8 · 28/27	16	19
2A.	10/9 · 21/20 · 8/7	18	21
2B.	21/20 · 10/9 · 8/7	19	22
3A.	16/15 · 9/8 · 10/9	13	16
3B.	10/9 · 9/8 · 16/15	13	16
4A.	256/243 · 9/8 · 9/8	19	22
5.	10/9 · 27/25 · 10/9	17	20

5-33. *The most consonant genera according to Euler's gradus suavitatis function. These ratios are the most consonant permutations of the most consonant tunings of each of the genera. In cases where the most consonant permutation according to Barlow's functions is different from the one(s) according to Euler's, both are given. The gradus suavitatis of a set of ratios is the GS of their least common multiple after the set has been transformed into a harmonic series.*

This failure, however, is a feature shared by the other simple theories of consonance based upon the prime factorization of intervals. Helmholtz's beat theory (Helmholtz [1877] 1954) and the semi-empirical "critical band" theories of Plomp and Levelt (1965) and Kameoka and Kuriyagawa (1969a, 1969b) avoid predicting infinite dissonance for mistuned consonances, but are more complex and difficult to use. The prime factor theories are adequate for theoretical work and for choosing between ideally tuned musical structures.

Statistical measures on tetrachordal space

The concepts of the degree of intervallic inequality and of the perceptual differences between tetrachords may be clarified by computing some of the standard statistical measures on a set of representative tetrachords. The arithmetic mean of the three intervals is 500/3 or 166.667 cents in equal temperament or $\sqrt[3]{(4/3)}$ in just intonation. The mean deviation, standard deviation, and variance are calculated according to the usual formulae for entire populations with $n = 3$. These data are shown in 5-34 for some representative tetrachords in just intonation and in 5-35 for a corresponding set in equal temperament. While not distinguishing permutations, these functions differentiate between genera quite well, although the degree to which the mathematical differences correlate with the perceptual is not known.

The geometric mean, harmonic mean, and root mean square (or quadratic mean) may be calculated in a similar fashion. Like the other statistical measures above, these are non-linear functions of the relative sizes of the intervals and they have considerable ability to discriminate between the various genera. The relevant data are shown in 5-36 and 5-37.

Several properties of these functions are apparent: for a given degree of intervallic asymmetry, the root mean square will show the greatest value,

5-34. *Mean deviations, standard deviations, and variances of the intervals of tetrachords in just intonation. The arithmetic mean has the constant value 166.67 cents (500/3) for all genera. In just intonation its value is the cube root of 4/3. The standard deviation and variance are computed with* n=3.

	MEAN DEV.	STANDARD DEV.	VARIANCE
28/27 · 36/35 · 5/4	146.87	155.88	24299.31
28/27 · 15/14 · 6/5	99.75	108.29	11725.73
25/24 · 16/15 · 6/5	99.75	107.12	11474.97
22/21 · 12/11 · 7/6	67.24	76.84	5904.95
16/15 · 9/8 · 10/9	36.19	39.38	1550.44
12/11 · 11/10 · 10/9	10.93	12.99	168.70

5-35. *Mean deviations, standard deviations, and variances of the intervals of tempered tetrachords.*

	MEAN DEV.	STANDARD DEV.	VARIANCE
ENHARMONIC (50 + 50 + 400)	155.56	164.99	27222.22
1:2 CHROMATIC (67 + 133 + 300)	88.89	98.13	9629.62
INTENSE CHROMATIC (100 + 100 + 300)	88.89	94.28	8888.89
SOFT DIATONIC (100 + 150 + 250)	55.56	62.36	3888.89
INTENSE DIATONIC (100 + 200 + 200)	44.44	47.14	2222.22
EQUAL DIATONIC	0.0	0.0	0.0

5-36. *Geometric mean, harmonic mean, and root mean square of the intervals of tetrachords in just intonation. For* n = 3, *the geometric mean is the cube root of* a·b·(500 − a − b)*; the harmonic mean is* 3/Σ (1/i), *where* 1/i = 1/a, 1/b, *and* 1/(500 − a − b)*; the root mean square is* $\sqrt{(\Sigma(i^2)/3)}$, *where* $i^2 = a^2, b^2, (500 - a - b)^2$.

	GEOMETRIC	HARMONIC	RMS
28/27 · 36/35 · 5/4	105.86	76.97	227.73
28/27 · 15/14 · 6/5	133.40	109.40	198.21
25/24 · 16/15 · 6/5	135.58	114.21	197.58
22/21 · 12/11 · 7/6	147.90	131.57	182.94
16/15 · 9/8 · 10/9	160.77	155.15	170.62
12/11 · 11/10 · 10/9	165.51	165.01	166.52

5-37. *Geometric mean, harmonic mean, and root mean square of tempered tetrachords.*

	GEOMETRIC	HARMONIC	RMS
ENHARMONIC (50 + 50 + 400)	100.0	70.59	234.52
1:2 CHROMATIC (67 + 133 + 300)	138.79	116.38	193.41
INTENSE CHROMATIC (100 + 100 + 300)	144.23	128.57	191.41
SOFT DIATONIC (100 + 150 + 250)	155.36	145.16	177.95
INTENSE DIATONIC (100 + 200 + 200)	158.74	150.0	173.21
EQUAL DIATONIC	166.67	166.67	166.67

the geometric the next, and the harmonic the least, except for the arithmetic mean, which is insensitive to this parameter.

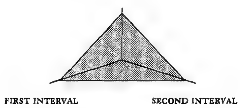
The set of all possible tetrachords instead of just representative examples or selected pairs may be studied by computing these standard statistical measures over the whole of tetrachordal space. This space may be defined by magnitudes of the first and second intervals (parhypate to hypate and lichanos to parhypate) as the third interval (mese to lichanos) is completely determined by the values of the first two.

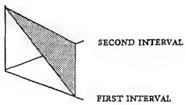
This idea may be made clearer by plotting a simple linear function such as the third tetrachordal interval itself versus the first and second intervals. The third interval may be defined as $500 - x - y$, where x is the lowest interval and y the second lowest. The domain of this function is defined by the inequalities $0 \le x \le 500$ cents, $0 \le y \le 500$ cents, and $x + y \le 500$ cents. 5-38 depicts the "third interval function" from two angles. Its values range from 0 to 500 cents.

The arithmetic, geometric, harmonic, and root mean square functions are shown in 5-39 through 5-41. The arithmetic mean is a plane of constant height at 166.667 cents for all values of the three intervals. The geometric and harmonic means have dome and arch shapes respectively, while the root mean square somewhat resembles the roof of a pagoda. The shapes of these latter means may be clearer in the contour plots in the lower portions of the figures.

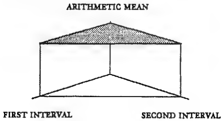
One may conclude that the arithmetic mean obscures the apparent distance between genera, the geometric mean reveals it, the harmonic mean maximizes it, and the root mean square exaggerates it. This conclusion is illustrated in 5-43 where a cross-section through the plot is made where the second interval has the value 166.667 cents and the first interval varies from

5-38. *The third interval function, seen frontally and obliquely. The three intervals are parhypate to hypate, lichanos to parhypate, and mese to lichanos. They always sum 500 cents (3/2 in just intonation).*





5-39. Arithmetic mean of the three tetrachordal intervals. The arithmetic mean has the constant value of 166.67 cents. The domain of this function is the x and y axes ($0 < x < 500$), ($0 < y < 500$), and the line $y = 500 - x$, where x and y are the first and second intervals of the tetrachord. The third interval may also approach zero.



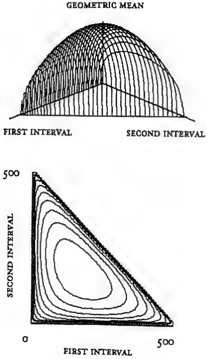
0 to 333.333 cents. The means are all equal when all three intervals of the tetrachord are 166.667 cents.

The analogous representation is applied to the mean deviation, standard deviation, and variance, which are shown in 5-44–46. The variance has been divided by 100 so that it may be plotted on the same scale as the other statistical functions.

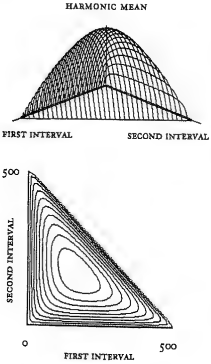
These functions have a minimum value of zero when all three intervals of the tetrachord are 166.667 cents each. This is seen most clearly in the cross-section plot of 5-47.

Based on its properties with respect to the four means and three statistical measures, the equally tempered division of the fourth appears to be a most interesting genus. It is the point where the three means are equal and where the statistical functions have their minima.

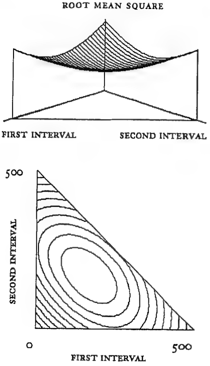
5-40. Geometric mean of the three tetrachordal intervals.



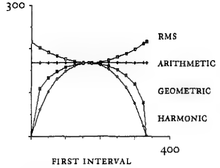
5-41. Harmonic mean of the three tetrachordal intervals.



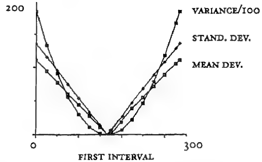
5-42. Root mean square of the three tetrachordal intervals.



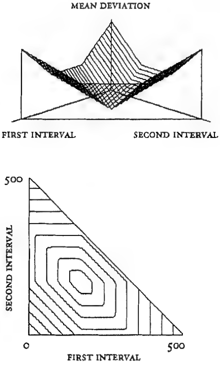
5-43. *Cross-sections of the various means of the three tetrachordal intervals when the second interval equals 166.67 cents.*



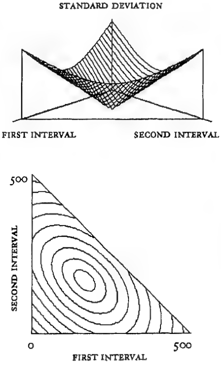
5-47. *Cross-section of the mean deviation, standard deviation, and variance of the three tetrachordal intervals when the second interval equals 166.67 cents.*



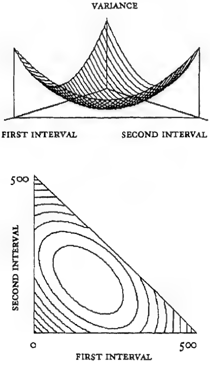
5-44. *Mean deviation of the three tetrachordal intervals.*



5-45. *Standard deviations of the three tetrachordal intervals.*



5-46. *Variance of the three tetrachordal intervals.*



5-48. *Interval sets of the abstract tetrachord, 0 a a+b 500. In just intonation the abstract tetrachord may be written 1/1 a a·b 4/3 or 0 a a+b 498 cents, and the intervals adjusted accordingly.*

SUCCESSIVE INTERVALS

0 a $a+b$ 500

a b $500-a-b$

POLANSKY SET

0 a $a+b$ 500

a $a+b$ 500

b $500-a$

$500-a-b$

DIFFERENCE SET

0 a $a+b$ 500

a b $500-a-b$

$b-a$ $500-a-2b$

$500-3b$

Polansky's morphological metrics

A more sophisticated approach with potentially greater power to discriminate between musical structures has been taken by Larry Polansky (1987b). While designed to handle larger and more abstract sets of elements than tetrachords, i.e., the type of scale and scale-like aggregates discussed in chapters 6 and 7, and even sets of timbral, temporal, or rhythmic information, Polansky's *morphological metrics* may be applied to smaller formations as well.

Morphological metrics are distance functions computed on the notes or intervals between the notes of an ordered musical structure. A morphological metric is termed linear or combinatorial according to the number of elements or intervals used in the computations: the more intervals or elements used in the computation, the more combinatorial the metric. In other words, combinatorial metrics tend to take into account more of the relationships between component parts. A strictly linear interval set as well as two of the possible combinatorial interval sets derived from an abstract, generalized tetrachord are shown in 5-48. For a strictly linear interval set of a morphology (or scale) of length L, there are $L-1$ intervals. The maximum combinatorial length for a morphology of length L is the binomial coefficient $(L^2-L)/2$, notated as L_m.

The simplest of Polansky's metrics is the ordered linear absolute magnitude (OLAM) metric which is the average of the absolute value of differences between corresponding members of two tetrachords. In the case of two tetrachords spanning perfect fourths of 500 cents, this function reduces to the sum of the absolute values of the differences between the two parhypatai and the two lichanoi divided by four. Given two tetrachords $a_1 + b_1 + 500 - a_1 - b_1$ and $a_2 + b_2 + 500 - a_2 - b_2$, the equation is:

$$\sum_{i=2}^{L} | e_{1_i} - e_{2_i} | / L,$$

where $L = 4$ and $e_{n_i} = (0, a_1, a_1 + b_1, 500)$ cents and $(0, a_2, a_2 + b_2, 500)$ cents. When not divided by L, this metric is identical to the Minkowski or "city block" metric previously discussed. Note that the OLAM metric does not take intervals into account, so it looks at L rather than $L-1$ values.

A simpler formula, $(|a_2 - a_1|$ and $|a_2 + b_2 - a_1 - b_1|) / 2$, would be defensible in this context as zero and 500 cents are constant for all tetrachords of this type. If the tetrachords are built above different tonics or their

fourths spanned different magnitudes, i.e., 500 and 498 or 583, etc., the first equation must be used.

The next simplest applicable metric is the ordered linear intervallic magnitude (OLIM) metric which is the average of the absolute values of the difference between the three intervals which define the tetrachords. In the case of the two tetrachords above, the intervals are a_1, b_1, $500-a_1-b_1$ and a_2, b_2, $500-a_2-b_2$. The equation for this metric function is:

$$\left| \sum_{i=2}^{L} (| e_{1_i} - e_{1_{i-1}} | - | e_{2_i} - e_{2_{i-1}} |) \right| / (L-1), \; L-1=3,$$

where i ranges from 2 through L, since intervals are being computed.

In 5-49, these two simple metrics are applied to a group of representative tetrachords in just intonation. The melodically similar tempered cases are shown in 5-50. Permutations of genera are analyzed in 5-51 and 5-52. The OLAM metric distinguishes between these genera quite well; the OLIM less so, but patterns are suggested which data on a larger set of tetrachords

5-49. *Ordered linear absolute magnitude (upper) and ordered linear intervallic magnitude (lower) metrics on tetrachords in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	17.67 47.11	19.60 47.11	34.25 79.63	63.17 135.94	72.90 135.94
28/27 · 15/14 · 6/5		1.93 5.14	16.59 32.51	45.50 88.83	55.23 88.83
25/24 · 16/15 · 6/5			14.66 32.51	43.57 88.83	53.30 88.83
22/21 · 12/11 · 7/6				28.92 56.31	38.64 56.31
16/15 · 9/8 · 10/9					9.73 25.94

5-50. *Ordered linear absolute magnitude (upper) and ordered linear intervallic magnitude (lower) metrics on tempered genera.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	29.17 66.67	37.50 66.67	50.0 100.0	62.50 133.33	87.50 155.56
1:2 CHROMATIC (67 + 133 + 300)		8.33 22.22	20.83 33.33	33.33 66.67	58.33 88.89
INTENSE CHROMATIC (100 + 100 + 300)			8.333 33.33	25.0 66.67	50.0 88.89
SOFT DIATONIC (100 + 150 + 250)				12.50 33.33	37.50 55.56
INTENSE DIATONIC (100 + 200 + 200)					25.0 44.44

5-51. *Ordered linear absolute magnitude (upper) and ordered linear intervallic magnitude (lower) metrics on Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	84.39 225.03	84.39 225.03	3.55 9.46	165.22 225.03	161.68 215.57
28/27 · 5/4 · 36/35		7.10 9.46	87.93 225.03	80.83 215.57	84.38 225.03
36/35 · 5/4 · 28/27			80.83 215.57	87.93 225.03	84.39 225.03
36/35 · 28/27 · 5/4				225.03 225.03	165.22 225.03
5/4 · 28/27 · 36/35					3.55 9.46

5-52. *Ordered linear absolute magnitude (upper) and ordered linear intervallic magnitude (lower) metrics on permuted tempered tetrachords.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	87.50 233.3	175.0 233.3
50 + 400 + 50		87.50 233.3

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	50.0 133.3	100.0 133.3
100 + 300 + 100		50.0 133.3

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	25.0 66.67	50.0 66.67
200 + 100 + 200		25.0 66.67

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	25.0 66.67	12.50 33.33	50.0 100.0	62.50 100.0	75.0 100.0
100 + 250 + 150		37.50 100.0	25.0 33.33	37.50 100.0	50.0 100.0
150 + 100 + 250			37.50 100.0	50.0 66.67	62.50 100.0
150 + 250 + 100				37.50 100.0	25.0 66.67
250 + 100 + 150					12.50 33.33

may reveal. In particular, the OLIM metric fails to distinguish between permutations of tempered tetrachords.

In theory, morphological metrics on combinatorial interval sets have greater discriminatory power than metrics on linear sets. Two sets of combinatorial intervals were derived from the simple successive intervals of 5-48. The first set, the Polansky set, is that described by Polansky (1987b). The second set, the difference set, was constructed from iterated differences of differences (Polansky, personal correspondence).

The ordered combinatorial intervallic magnitude (OCIM) metric is the average of the absolute value of the differences between corresponding elements of the musical structure. Its definition is:

$$\sum_{j=1}^{L-1} \sum_{i=1}^{L-j} |\Delta(e_{1_i}, e_{1_{i+j}}) - \Delta(e_{2_i}, e_{2_{i+j}})| / L_m,$$

where L_m = the number of intervals in the set (the binomial coefficient, described above). To apply it to other combinatorial interval sets, it must be appropriately modified to something like:

$$\sum_{i=2}^{L} |(I_{1_i} - I_{2_i})| / L_m,$$

where I_{n_i} are the elements of a set like the difference set of 5-48.

As can be seen in 5-53 and 5-54, the OCIM metric calculated on the two sets of intervals from these tetrachords discriminates between genera very well. Both sets of intervals are roughly equivalent with this metric.

Permutations are studied in 5-55 and 5-56. On neither interval set does the OCIM metric distinguish permutations completely.

5-53. *Ordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from tetrachords in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	35.34 94.23	36.62 86.52	62.65 141.68	110.08 223.11	116.57 184.20
28/27 · 15/14 · 6/5		3.86 10.28	27.31 47.45	74.75 128.88	81.23 104.01
25/24 · 16/15 · 6/5			26.03 55.16	15.36 136.59	79.94 106.58
22/21 · 12/11 · 7/6				47.43 81.43	53.92 61.10
16/15 · 9/8 · 10/9					19.45 51.87

5-54. *Ordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from tempered tetrachords.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	52.78 116.67	58.33 83.33	83.33 150.0	108.33 216.67	136.11 194.44
1:2 CHROMATIC (67 + 133 + 300)		16.67 44.44	30.56 38.89	55.56 100.0	83.33 100.0
INTENSE CHROMATIC (100 + 100 + 300)			25.0 66.67	50.0 136.33	77.78 111.11
SOFT DIATONIC (100 + 150 + 250)				25.0 66.67	52.78 61.11
INTENSE DIATONIC (100 + 200 + 200)					38.39 55.56

5-55. *Ordered combinatorial intervallic magnitude metric on Polansky (upper) and difference (lower) interval sets on permutations of Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	168.77 450.06	168.77 450.06	7.10 18.92	222.66 229.76	215.57 215.57
28/27 · 5/4 · 36/35		9.46 9.46	171.14 435.87	161.68 431.14	168.77 450.06
36/35 · 5/4 · 28/27			161.68 431.14	171.14 435.87	168.77 450.06
36/35 · 28/27 · 5/4				225.03 225.03	222.66 229.76
5/4 · 28/27 · 36/35					7.10 18.92

5-56. *Ordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from permuted tempered tetrachords.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	175.0 466.67	233.33 233.33
50 + 400 + 50		175.0 466.67

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	100.0 266.67	133.33 133.33
100 + 300 + 100		100.0 266.67

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	50.0 133.33	66.67 66.67
200 + 100 + 200		50.0 133.33

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	50.0 133.33	25.0 66.67	83.33 150.0	91.67 116.67	100.0 100.0
100 + 250 + 150		75.0 200.0	33.33 33.33	75.0 200.0	83.33 150.0
150 + 100 + 250			75.0 200.0	66.67 66.67	91.67 116.67
150 + 250 + 100				75.0 200.0	50.0 133.33
250 + 100 + 150					25.0 66.67

Unordered counterparts of the ordered metrics are also defined. Although the unordered linear absolute or intervallic magnitude metrics are of little use in this context, the unordered combinatorial intervallic magnitude (UCIM) metric is rather interesting when computed on these two interval sets.

For the Polansky interval set, the metric is:

$$\left|\sum_{j=1}^{L-1} \sum_{i=1}^{L-j} \Delta(e_{1i}, e_{1i+j})/L_m - \sum_{j=1}^{L-1} \sum_{i=1}^{L-j} \Delta(e_{2i}, e_{2i+j})/L_m\right|, L_m=6.$$

This function is the absolute value of the difference between the averages of the corresponding intervals. For the difference set, the formula becomes:

$$\left|\sum_{i=2}^{L} (I_{1i})/L_m - \sum_{i=2}^{L} (I_{2i})/L_m\right|, L_m=6,$$

where the I_{ni} are the elements of the set.

5-57 and 5-58 show the data for the same group of tetrachords as before. Genera are fairly well discriminated by this metric, especially when calculated on the Polansky interval set, but not as well with the difference set intervals. Neither are particularly successful for distinguishing permutations with this metric (5-59 and 5-60).

5-57. *Unordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from tetrachords in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	11.78 47.11	10.49 44.54	16.98 73.77	25.86 119.68	19.37 106.71
28/27 · 15/14 · 6/5		1.29 2.57	5.20 26.65	14.08 72.57	7.59 59.60
25/24 · 16/15 · 6/5			6.48 29.23	15.36 75.14	8.88 62.17
22/21 · 12/11 · 7/6				8.88 45.91	2.39 32.94
16/15 · 9/8 · 10/9					6.48 12.97

5-58. *Unordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from tempered tetrachords.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC 50 + 50 + 400	13.889 61.11	8.333 50.0	16.67 83.33	25.0 116.67	19.44 116.67
1:2 CHROMATIC 67 + 133 + 300		5.556 11.11	2.778 22.22	11.11 55.56	5.556 55.56
INTENSE CHROMATIC 100 + 100 + 300			8.333 33.33	16.67 66.67	11.11 66.67
SOFT DIATONIC 100 + 150 + 250				8.333 33.33	2.778 33.33
INTENSE DIATONIC 100 + 200 + 200					5.556 0.0

5-59. *Unordered combinatorial intervallic magnitude metric on Polansky (upper) and difference (lower) interval sets on permutations of Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	56.26 225.03	56.26 220.30	2.36 4.73	2.36 117.24	0.0 107.78
28/27 · 5/4 · 36/35		0.0 4.73	53.89 220.30	53.89 107.78	56.26 117.24
36/35 · 5/4 · 28/27			53.89 215.57	53.89 103.05	56.26 112.51
36/35 · 28/27 · 5/4				0.0 112.51	222.66 103.05
5/4 · 28/27 · 36/35					2.36 9.46

5-60. *Unordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from permuted tempered tetrachords.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	58.33 233.33	0.0 116.67
50 + 400 + 50		58.33 116.67

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	33.33 133.33	0.0 66.67
100 + 300 + 100		33.33 66.67

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	16.67 33.33	0.0 33.33
200 + 100 + 200		16.67 66.67

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	16.67 66.67	8.333 16.67	16.67 83.33	8.333 16.67	0.0 50.0
100 + 250 + 150		25.0 83.33	0.0 16.67	25.0 50.0	16.67 16.67
150 + 100 + 250			25.0 100.0	0.0 33.33	8.333 66.67
150 + 250 + 100				25.0 66.67	16.67 33.33
250 + 100 + 150					8.333 33.33

In addition to absolute and intervallic metrics, directional metrics are also defined. Directional metrics measure only the contours of musical structures, i.e., whether the differences between successive elements are positive, negative or zero. Although these metrics are perhaps the most interesting of all, they are generally inapplicable to tetrachords because tetrachords are sets of four monotonically increasing pitches whose differences are always positive (or negative if the tetrachord is presented in descending order). Directional metrics, however, are very applicable to melodies constructed from the notes of tetrachords or from tetrachordally derived scales such as those of chapter 6.

The intervals of the tetrachordal difference set, however, are not necessarily monotonic and therefore combinatorial directional metrics may be computed on these intervals. Two such metrics were calculated for the same set of tetrachords and permutations used above, the ordered

5-61. *Ordered (upper) and unordered (lower) combinatorial interval direction metrics on difference sets from tetrachords in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	.1667 .3333	.1667 .3333	.1667 .3333	.5000 .3333	.1667 .3333
28/27 · 15/14 · 6/5		0.0 0.0	0.0 0.0	.3333 .6667	0.0 0.0
25/24 · 16/15 · 6/5			0.0 0.0	.3333 .6667	0.0 0.0
22/21 · 12/11 · 7/6				.3333 .667	0.0 0.0
16/15 · 9/8 · 10/9					.5000 .3333

5-62. *Ordered (upper) and unordered (lower) combinatorial interval direction metrics on difference sets from tempered genera.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	.1667 .3333	0.0 0.0	.1667 .3333	.5000 .3333	.3333 .6667
1:2 CHROMATIC (67 + 133 + 300)		.1667 .3333	0.0 0.0	.3333 .6667	.5000 1.00
INTENSE CHROMATIC (100 + 100 + 300)			.1667 .3333	.5000 .3333	.3333 .6667
SOFT DIATONIC (100 + 150 + 250)				.3333 .6667	.5000 1.00
INTENSE DIATONIC (100 + 200 + 200)					.3333 .3333

combinatorial intervallic directional (OCID) metric and its unordered counterpart, the unordered combinatorial intervallic directional (UCID) metric. The OCID metric is the average of the differences of the signs of corresponding intervals. The sign (sgn) of an interval is -1, 0, or +1 according to whether the interval is decreasing, constant or increasing. The **difference** (**diff**) is 1 when the signs are dissimilar, otherwise the difference is zero. The definition of the OCID metric on the difference set is:

$$\sum_{i=2}^{L} \mathrm{diff}(\mathrm{sgn}(I_{1i}), \mathrm{sgn}(I_{2i}))/L_m, L_m = 6.$$

The UCID metric is the average of the absolute values of the numbers of intervals with each sign. The definition of UCID on the difference set is:

$$\sum_{i=2}^{L} \mid \#e_1^v - \#e_2^v \mid)/L_m, L_m = 6,$$

where $\#e_n^v$ = the number of intervals in the matrix such that $v = \mathrm{sgn}(I_{ni})$; i.e., $v = [-1, 0, 1]$.

The data from these computations are shown in 5-61 and 5-62. Similar results were obtained with tetrachordal permutations (5-63 and 5-64).

5-63. *Ordered (upper) and unordered (lower) combinatorial interval direction metrics on difference sets from permutations of Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	.5000 .3333	.5000 .3333	.1667 .3333	.1667 .3333	0.0 0.0
28/27 · 5/4 · 36/35		0.0 0.0	.3333 .6667	.3333 0.0	.5000 .3333
36/35 · 5/4 · 28/27			.3333 .6667	.3333 0.0	.5000 .3333
36/35 · 28/27 · 5/4				.3333 .6667	.1667 .3333
5/4 · 28/27 · 36/35					.1667 .3333

5-64. *Ordered (upper) and unordered (lower) combinatorial interval direction metrics on difference sets from permuted tempered tetrachords.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	.5000 .6667	.3333 .3333
50 + 400 + 50		.5000 .3333

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	.5000 .6667	.3333 .3333
100 + 300 + 100		.5000 .3333

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	.5000 .3333	.3333 .3333
200 + 100 + 200		.5000 .6667

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	.3333 .6667	.1667 .3333	.3333 .6667	.1667 .3333	.3333 .6667
100 + 250 + 150		.5000 .3333	0.0 0.0	.5000 .3333	.3333 0.0
150 + 100 + 250			.5000 .3333	0.0 0.0	.1667 .3333
150 + 250 + 100				.5000 .3333	.3333 0.0
250 + 100 + 150					.1667 .3333

Rothenberg propriety

David Rothenberg has developed criteria derived from the application of concepts from artificial intelligence to the perception of pitch (Rothenberg 1969, 1975, 1978; Chalmers 1975, 1986b). In Rothenberg's own words (personal communication): "These concepts relate the intervallic structure of scales to the perceptibility of various musical relations in music using these scales. Only the relative sizes of the intervals between scale tones, not the precise sizes of these intervals are pertinent." These concepts are applicable to scales of any cardinality whether or not the intervals repeat at some interval of equivalence. In practice, most scales repeat at the octave, though cycles of tetrachords and pentachords are found in Greek Orthodox liturgical music (Xenakis 1971; Savas 1965).

To apply Rothenberg's concepts, the first step is to construct a difference matrix from the successive intervals of an n-tone scale. The columns of the matrix are the intervals measured from each note to every other one of the scale. The rows t_n of the matrix are the sets of adjacent intervals measured from successive tones. These intervals are defined conventionally: the row of seconds (t_1) comprises the differences between adjacent notes; the row of thirds (t_2) consists of the differences between every other note; etc., up to the interval of equivalence (t_n). Row t_0 contains the original scale.

A number of functions may be calculated on this matrix. The most basic of these is *propriety*. A scale is *strictly proper* if for all rows every interval in row t_{n-1} is less than every interval in row t_n. If the largest interval in any row t_{n-1} is at most equal to the smallest interval in row t_n, the scale is termed *proper*. These equal intervals are considered ambiguous as their perception depends upon their context. A familiar example is the tritone (F–B in the C major mode in 12-tone equal temperament), which may be perceived as either a fourth or a fifth.

Scales with overlapping interval classes, i.e., those with intervals in rows t_{n-1} larger than those in rows t_n, are *improper*. These contradictory intervals tend to confound one's perception of the scale as a musical entity, and improper scales tend to be perceived as collections of principal and ornamental tones. Improper scales may contain ambiguous intervals as well.

5-65 illustrates these concepts with certain tetrachordal heptatonic scales in the 12- and 24-tone equal temperaments. The first example is the intense diatonic of Aristoxenos. The scale is proper and the tritone is ambiguous. The second scale is Aristoxenos's soft diatonic which is also

proper, but replete with ambiguous intervals. A composer using this scale might prefer to fix the tonic with drone or restrict modulation so as to avoid exposing the ambiguous intervals. The next scale is patterned after certain common Islamic scales employing modally neutral intervals. It is strictly proper, a feature it shares with the more familiar five-note black key scale in 12-tone equal temperament.

The final two examples, Aristoxenos's intense chromatic and his enharmonic, are improper. The majority of the intervals of these scales are either ambiguous or contradictory. These scales are most likely to be heard and used as pentatonic sets with alternate tones or inflections.

Because the major (0 400 700 cents, 4:5:6 in just intonation), minor (0 300 700 cents, 10:12:15), subminor (0 250 700 cents, 6:7:9), and supramajor (0 450 700 cents, 14:18:21) triads are strictly proper, they can serve

5-65. *Rothenberg difference matrices. The row index is t_n. Max (t_n) is the largest entry in row t_n. Min (t_n) is the smallest entry in row t_n. The intense diatonic tetrachord is 1 + 2 + 2 degrees or 6 + 12 + 12 parts. The soft diatonic derives from 2 + 3 + 5 or 6 + 9 + 15 parts. The neutral diatonic is 3 + 4 + 3 degrees, a permutation of 9 + 9 + 12 parts. The intense chromatic is 1 + 1 + 3 degrees. The enharmonic tetrachord is 1 + 1 + 8 degrees. Intervals in parentheses are ambiguous; those in square brackets are contradictory.*

INTENSE DIATONIC IN 12-TONE ET: PROPER

t_0	0	2	4	6	7	9	11	12/0
t_1	1	2	2	2	1	2	2	MAX (t_3) = MIN (t_4) = 6
t_2	3	4	4	3	3	4	3	
t_3	5	(6)	5	5	5	5	5	
t_4	7	7	7	7	(6)	7	7	
t_5	8	9	9	8	8	9	9	
t_6	10	11	10	10	10	11	10	
t_7	12	12	12	12	12	12	12	

SOFT DIATONIC IN 24-TONE ET: PROPER

t_0	0	2	5	10	14	16	19	24/0
t_1	2	3	(5)	4	2	3	(5)	MAX (t_1) = MIN (t_2)
t_2	(5)	8	(9)	6	(5)	8	(5)	MAX (t_2) = MIN (t_3)
t_3	10	(12)	11	(9)	10	10	10	MAX (t_3) = MIN (t_4)
t_4	14	14	14	14	(12)	13	(15)	MAX (t_4) = MIN (t_5)
t_5	16	17	(19)	16	(15)	18	(19)	ETC.
t_6	(19)	22	21	(19)	20	22	21	
t_7	24	24	24	24	24	24	24	

NEUTRAL DIATONIC IN 24-TONE ET: STRICTLY PROPER

t_0	0	3	7	10	14	17	21	24/0
t_1	3	4	3	4	3	4	3	MAX (t_{n-1}) < MIN (t_n)
t_2	7	7	7	7	7	7	6	
t_3	10	11	10	11	10	10	10	
t_4	14	14	14	14	13	14	13	
t_5	17	18	17	17	17	17	17	
t_6	21	21	20	21	20	21	20	
t_7	24	24	24	24	24	24	24	

INTENSE CHROMATIC IN 12-TONE ET: IMPROPER

t_0	0	1	2	5	7	8	9	12/0
t_1	1	1	[3]	[2]	1	1	[3]	MAX (t_1) > MIN (t_2)
t_2	[2]	4	[5]	3	[2]	4	4	MAX (t_2) > MIN (t_3)
t_3	5	(6)	(6)	[4]	5	5	5	
t_4	7	7	7	7	(6)	(6)	[8]	
t_5	8	8	10	8	[7]	9	10	
t_6	9	11	11	9	10	11	11	
t_7	12	12	12	12	12	12	12	

ENHARMONIC IN 24-TONE ET: IMPROPER

t_0	0	1	2	10	14	15	16	24/0
t_1	1	1	[8]	4	1	1	[8]	MAX (t_1) > MIN (t_2)
t_2	[2]	9	[12]	5	[2]	9	9	MAX (t_2) > MIN (t_3)
t_3	10	[13]	[13]	[6]	10	10	10	MAX (t_3) > MIN (t_4)
t_4	14	14	14	14	[11]	[11]	[18]	
t_5	15	15	22	15	[12]	19	22	
t_6	16	23	23	16	20	23	23	
t_7	24	24	24	24	24	24	24	

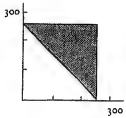
5-66. *Propriety limits of tetrachords. The differences are in cents and an underlying zero modulo 12 equal temperament is assumed. The results for just intonation are virtually identical except that the fourth of 498.045 cents and a whole tone of 203.91 cents replace the 500- and 200-cent intervals in the computations.*

ROWS	DIFFERENCE MATRIX		
t_1	a	b	$500 - a - b$
t_2	$a + b$	$500 - a$	$500 - b$
t_3	500	500	500

CONSTRAINTS: $0 < a < 250$; $0 < b < 250$; $250 < a + b < 500$.

VERTICES: 0, 250; 250, 0; 250, 250.

5-67. *Propriety limits for isolated tetrachords and conjunct chains of tetrachords.*



as sets of principal tones for improper scales. The various sets of principal tones would be used as the main carriers of melodies, while the auxiliary tones would be used as ornaments. This topic deserves more extended discussion than is appropriate here and Rothenberg's original papers should be consulted (Rothenberg 1969, 1975, 1978).

The fact that the minor and septimal minor triads are strictly proper may explain certain musically significant cadential formulae in the Dorian modes of the enharmonic and chromatic genera. These consist of a downward leap from the octave to the lowered submediant (trite), then down to the subdominant (mese) before ending up on the dominant (paramese). This formula may be repeated a fifth lower, beginning with a leap from the subdominant (mese) to the lowered supertonic (parhypate) and then down to the *subtonic* (hyperhypate) before ending on hypate (chapters 6 and 7). Minor triads are outlined in the chromatic genus and septimal minor triads in the enharmonic. The latter chords contain the important interval of five dieses called eklysis by the Greek theorists, and in fact, the jump from parhypate to hyperhypate is seen in the *Orestes* fragment (Winnington-Ingram 1936). The upper submediants (lichanos and paranete) may be substituted in both genera; the major triad appearing in the chromatic genus is also strictly proper.

As has been seen above, the propriety criterion separates those scales derived from chromatic and enharmonic tetrachords from those generated by diatonic genera. As will be seen later, the situation is somewhat more complex; under certain conditions, some diatonic tetrachords yield only improper scales, while some chromatic genera can combine with diatonic tetrachords to generate proper mixed heptatonic scales.

Propriety may be computed for abstract classes of scales or subscalar modules rather than for specific instances by replacing one or more of the intervals by variables. If the three subintervals of the tetrachord are written as a, b, and $500 - a - b$ (a, b, and $4b/3a$ in just intonation), one can calculate the Rothenberg difference matrix and determine the propriety limits for isolated tetrachords or conjunct chains where the interval of equivalence is the fourth. Such chains were present in the earlier stages of classical Greek music and are still extant in contemporary Greek Orthodox liturgical music (chapter 6 and Xenakis 1971).

The computation is performed by solving the inequalities formed by setting each of the elements of rows t_n less than each of those in rows t_{n+1}.

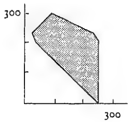
5-68. *Propriety limits of pentachords.*

ROWS	DIFFERENCE MATRIX			
t_1	a	b	$500 - a - b$	200
t_2	$a + b$	$500 - a$	$700 - a - b$	$200 + a$
t_3	500	$700 - a$	$700 - b$	$200 + a + b$
t_4	700	700	700	700

CONSTRAINTS: $0 < a < 250$; $0 < b < 250$; $250 < a + b < 500$; $2a + b < 700$; $a + 2b < 700$; $b - a < 200$; $300 < 2a + b$.

VERTICES: 250, 0; 50, 200; 33.3, 233.3; 100, 300; 233.3, 233.3; 250, 200.

5-69. *Propriety limits for isolated pentachords and conjunct chains of pentachords.*



In practice, the work may be minimized because only the elements in the first $(n + 1) / 2$ rows of an n-tone scale need be considered. One may also ignore relations that are tautological when all the intervals are positive.

The result is a set of constraints on the sizes of intervals a and b, shown in 5-66. Tetrachords and conjunct chains of tetrachords spanning perfect fourths, are strictly proper when intervals a and b satisfy these constraints. The tetrachords and chains are proper when their intervals equal the extrema of the constraints. For values outside these limits, the tetrachords and conjunct chains are improper.

Because the three intervals a, b, and $500 - a - b$ add to a constant value, there are only two degrees of freedom. Therefore, the domain over which tetrachords are proper may be displayed graphically in two dimensions. The region in the $a \cdot b$ plane within which tetrachords are strictly proper is shown in 5-67. The vertices define an area in the $a \cdot b$ plane within which the constraints are satisfied. Points on the edges of the triangular region correspond to proper tetrachords. The two points on the axes are also proper as *trichords*, which are degenerate tetrachords with only three notes.

Similarly, the propriety limits for pentachords consisting of a tetrachord and an annexed disjunctive tone (200 cents or 9/8) may be determined. The difference matrix is shown in 5-68. As all circular permutations of a scale have the same value for propriety, it is immaterial whether the disjunctive tone is added at the top or bottom of the tetrachord. The region satisfying the propriety constraints for isolated pentachords and pentachordal chains is shown in 5-68.

Similar calculations may be carried out for complete heptatonic scales consisting of two identical tetrachords and a disjunctive tone. This tone

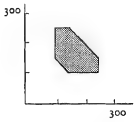
5-70. *Propriety limits for heptatonic scales with identical tetrachords.*

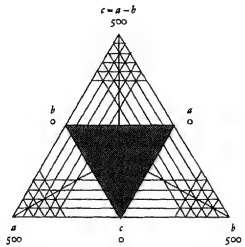
a	b	$500-a-b$	200	a	b	$500-a-b$
$a+b$	$500-a$	$700-a-b$	$200+a$	$a+b$	$500-a$	$500-b$
500	$700-a$	$700-b$	$200+a+b$	500	500	500
700	700	700	700	$500+a$	$500+b$	$1000-a-b$

CONSTRAINTS: $100 < a < 250$; $100 < b < 250$; $250 < a + b < 400$.

VERTICES: 100,150; 100,250; 150,100; 150,250; 250,150; 250,100.

5-71. *Propriety limits for heptatonic scales with identical tetrachords.*





5-72. Propriety limits for tetrachords and tetrachordal chains. These limits are for chains of conjunct tetrachords such as are found in Greek Orthodox liturgical music (Xenakis 1971).

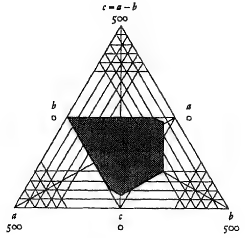
may be placed between the tetrachords or at either end to complete the octave (chapter 6). The results of the calculations are given in 5-70. The region of propriety is shown in 5-71.

Complete tetrachordal space

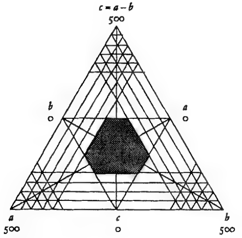
An alternative mode of graphic representation may be clearer. Physical chemists have long been accustomed to plotting phase diagrams for three component mixtures on equilateral triangle graphs. The three altitudes are interpreted as the fractions of each component in the whole mixture. There are only two degrees of freedom as the sum of the composition fractions must equal unity. The data from 5-66, 5-68, and 5-70 have been replotted in 5-72–73.

5-72 shows the range over which the intervals a, b, and $500 - a - b$ may vary and still result in proper tetrachords. Pentachords are shown in 5-73 and heptatonic scales in 5-74.

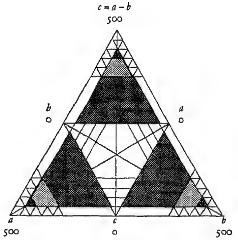
The advantage of the triangular graph over the conventional rectangular type is most evident with the heptatonic scales of 5-74. All points in the interior of the semi-regular hexagonal region correspond to strictly proper scales, while the edges are sets of intervals that define scales that are merely



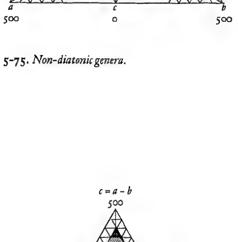
5-73. Propriety limits for pentachords and pentachordal chains.



5-74. Proper heptatonic scales.



5-75. *Non-diatonic genera.*



5-76. *Complete tetrachordal space.*

proper. The three triangular spaces lying between the long sides of the hexagon and the edge of the space contain diatonic genera which yield improper heptatonic scales. In certain cases to be discussed later, some of these tetrachords may be combined with other genera to produce proper mixed scales.

The six vertices of the central hexagon in 5-74 are the six permutations of the soft diatonic genus of Aristoxenos, 100 + 150 + 250 cents. The center of overall symmetry is the equal diatonic genus, 166.667 + 166.667 + 166.667 cents. The intersection of the altitudes of the triangle and the midpoints of the long sides of the hexagon are the three permutations of the intense diatonic, 100 + 200 + 200 cents, while the intersections with the midpoints of the short sides define the arrangements of the neo-Aristoxenian genus, 125 + 125 + 250 cents. This genus lies on the border of the chromatic and diatonic genera, but sounds chromatic because of the equal division of the pyknon.

The non-diatonic or pyknotic genera are portrayed in 5-75. The empty border around the filled regions delimits the commatic (25 cents) and subcommatic intervals. The small triangular regions in dark color near the vertices are the hyperenharmonic genera whose smallest intervals fall between 25 and 50 cents in this classification (see the neo-Aristoxenian classification above for more refined limits on the boundaries between the hyperenharmonic, enharmonic, and chromatic genera). Next are the trapezoidal enharmonic and chromatic zones which flank the unmarked central diatonic area. The enharmonic zone contains pyknotic intervals from 50 to 100 cents and the chromatic from 100 to 125 cents.

These data are summarized in 5-76. The diatonic tetrachords generating proper and strictly proper scales map into the central zone. The three triangular zones flanking the central region along the long sides of the hexagon are diatonic tetrachords which contain one of the small hyperenharmonic, enharmonic, or chromatic intervals. These diatonic genera yield improper scales. As in 5-75, the chromatic tetrachords lie in the large trapezoidal regions, with the enharmonic and hyperenharmonic beyond. The outer belts of the chromatic zones depict genera with enharmonic and hyperenharmonic intervals. Similarly, the enharmonic regions are divided into realms of pure enharmonic and enharmonic mixed with hyperenharmonic intervals.

Propriety of mixed scales

The computation of the propriety limits for heptatonic scales containing dissimilar tetrachords is a more complex problem. Since there are now four degrees of freedom, two for each of the tetrachords, the graphical methods used for the single tetrachord case are of limited use. It is possible, however, to consider the upper and lower tetrachords separately and to calculate absolute limits on the intervals of each. If a, b, and $500 - a - b$ are assigned to the intervals of the lower tetrachord and c, d, and $500 - c - d$ to the upper, one can compute the range of values for a and b over which it is possible to find an upper tetrachord with which a proper scale can be generated. Similar computations may be done for c and d. These results of these calculations are tabulated in 5-77 and are graphed in 5-78 and 5-79. These graphs use only those relations which are solely functions of a and b or c and d.

Triangular plots of the same data are depicted in 5-80 and 5-81. The union of the the upper and lower tetrachord regions corresponds to the pentachordal limits of 5-68 and 5-73, and their intersection is the proper diatonic region of 5-74. The upper and lower tetrachord regions are also the intervallic retrogrades of each other as propriety is unaffected by retrogression or circular permutation of the intervals.

The solution to the general case of finding the limits for mixed tetrachordal scales must satisfy all the inequalities that relate a, b, c, and d. It is difficult to display this four-dimensional solution space in two dimensions. One can, however, choose tetrachords from the lower or upper absolute

5-77. Propriety limits for heptatonic scales with mixed tetrachords. (Only the first four rows are shown.)

a	b	$500-a-b$	200	c	d	$500-c-d$
$a+b$	$500-a$	$700-a-b$	$200+c$	$c+d$	$500-c$	$500-c-d+a$
500	$700-a$	$700-a-b+c$	$200+c+d$	500	$500-c+a$	$500-c-d+a+b$
700	$700-a+c$	$700-a-b+c+d$	700	$500+a$	$500-c+a+b$	$1000-c-d$

CONSTRAINTS ON a AND b: $0 < a < 250$; $250 < a+b < 500$; $2a+b < 700$; $a+2b < 700$.

VERTICES: 100, 150; 100, 300; 250, 200; 250, 0; 233.3, 233.3.

CONSTRAINTS ON c AND d: $c < 250$; $250 < c+d < 400$; $d-c < 200$; $300 < 2c+d$.

VERTICES: 50, 200; 33.3, 233.3; 100, 300; 250, 150; 250, 0.

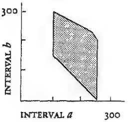
MUTUAL CONSTRAINTS ON a, b, c, AND d: $a < c+d$; $b < c+d$; $c < a+b$; $d < a+b$; $c < 2a$; $a+c < 500$; $b+c < 500$; $a+d < 500$; $b-c < 200$; $2c-a < 300$; $a-c < 100$; $c+d-a < 300$; $a+b+c < 700$; $2c+d-a < 500$; $c+2d-a < 500$; $a+b+d < 700$; $2a+2b-c < 700$; $a+b-c-d < 100$; $300 < a+c+d$; $c+d < 2a+b$; $200 < 2a+2b-c-d$; $2c+d-a-b < 300$; $2a-c-d < 500$; $200 < 2a+b-c$; $c+b+d-a < 500$; $500 < a+b+c+d$; $300 < 2c+2d-a$; $2a+b-2c-d < 200$.

propriety regions of 5-80 and 5-81 and find companion tetrachords which produce proper heptatonic scales when joined to them by a disjunctive tone. These computations are performed in the same way as in 5-70 and 5-77, except that the variables in one of the two tetrachords are replaced by the cents values of the intervals. The result of the calculations will be a range of values for the companion tetrachord.

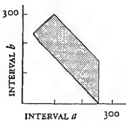
The three permutations of the intense diatonic genus in 12-tone equal temperament (100 + 200 + 200 cents, 200 + 100 + 200 cents, and 200 + 200 + 100 cents) as well as the neochromatic form of the syntonic chromatic (100 + 300 + 100 cents) were selected as lower tetrachords. The propriety limits for the upper companion tetrachords were then computed. These results are shown in 5-82.

Points in the interiors of the regions yield strictly proper scales, while those on the peripheries produce scales that are merely proper. The neochromatic tetrachord has only a one-dimensional solution space; the uppermost point corresponds to a mode of the harmonic minor scale.

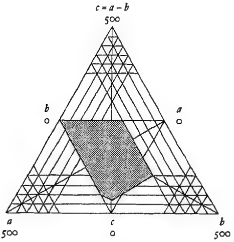
Similar calculations were performed for an additional 23 tetrachords and the results are tabulated in 5-83. In agreement with previous results (5-74 and 5-78), no proper scales could be formed from lower tetrachords whose first intervals were microtones.



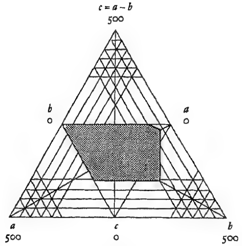
5-78. *Absolute propriety limits for lower tetrachords.*



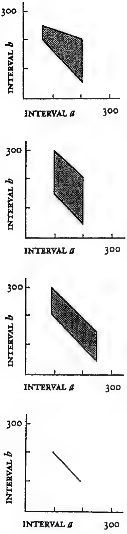
5-79. *Absolute propriety limits for upper tetrachords.*



5-80. *Absolute propriety limits for lower tetrachords.*



5-81. *Absolute propriety limits for upper tetrachords.*



5-82. *Propriety ranges for upper companion tetrachords: limits for the tetrachords (a) 100 + 200 + 200 cents, (b) 200 + 100 + 200 cents, (c) 200 + 200 + 100 cents, (d) 100 + 300 + 100 cents..*

Upper tetrachords may also be chosen and lower companion ranges subsequently calculated to yield scales that are the intervallic retrogrades or octave inversions of those above.

A number of interesting conclusions may be drawn from these data. Proper heptatonic tetrachordal scales containing microtones are only possible under certain conditions. The microtonal intervals may be present in either the upper or lower tetrachord provided they are not in the extreme positions, i.e., not intervals a or $500-c-d$.

Proper hexatonic scales also exist when tetrachordal intervals b or d equal zero and a and c are 250 cents. These scales may be analysed as containing a tetrachord, a disjunctive tone, and a trichord.

The tetrachordal genera which appear as vertices of the propriety regions are of great interest. In particular, the equal division 166.667 + 166.667 + 166.667 accepts as upper companions both chromatic and improper diatonic genera, including some with subcommatic intervals. Other new tetrachords occurring as vertices are the improper diatonic genera 33.333 + 233.333 + 233.333; this is very close to Al-Farabi's 49/48 · 8/7 · 8/7, and 50 + 250 + 200, which is approximated rather well by 40/39 · 52/45 · 9/8.

Work of other investigators

Several other investigators have independently developed descriptors functionally identical to Rothenberg's strict propriety. Gerald Balzano has used the notion of "coherence" in his work on microtonal analogs of the diatonic scale in 12-tone equal temperament (Balzano 1980). Though not tetrachordal, Balzano's scales are homologous to the tritriadic scales discussed in chapter 7. Ervin Wilson (personal communication) has applied the term *constant structure* to scales in which each instance of a given interval subtends the same number of subintervals, but not necessarily subintervals of the same magnitude or order. This property is also equivalent to propriety.

5-83. *Proper mixed tetrachord scales, in cents. These tetrachords can combine with a disjunctive tone and any tetrachord in the region defined by the vertices to yield proper or strictly proper scales. The retrogrades of these tetrachords can also serve as the upper tetrachords of proper scales. The third interval of each tetrachord may be found by subtracting the sum of the two tabulated intervals from 500 cents. The neo-chromatic tetrachord number 4 is the upper tetrachord of the harmonic minor mode. Its region of propriety is reduced to a line rather than an area in the tetrachordal interval plane. Tetrachords 11, 12, and 26 cannot form proper scales with any upper tetrachord.*

	LOWER TETRACHORD	VERTICES
1.	100 200 200	50, 200; 50, 250; 200, 200; 200, 50
2.	200 100 200	100, 150; 100, 300; 200, 200; 200, 50
3.	200 200 100	100, 200; 100, 300; 250, 150; 250, 50
4.	100 300 100	100, 200; 200, 100
5.	100 150 250	50, 250; 50, 200; 150, 150; 150, 100
6.	100 250 150	100, 150; 100, 250; 200, 150; 200, 50
7.	150 100 250	50, 200; 50, 250; 150, 150; 150, 100
8.	150 250 100	100, 275; 100, 200; 150, 250; 225, 175; 225, 75
9.	250 100 150	150, 150; 150, 250; 250, 150; 250, 50
10.	250 150 100	150, 150; 150, 250; 250, 150; 250, 50
11.	50 250 200	NO PROPER SCALES
12.	50 200 250	NO PROPER SCALES
13.	200 50 250	100, 150; 100, 200; 150, 150; 150, 100
14.	200 250 50	200, 150; 200, 200; 250, 150; 250, 100
15.	250 50 200	150, 150; 150, 250; 200, 200; 200, 100
16.	250 200 50	200, 150; 200, 200; 250, 150; 250, 100
17.	125 125 250	50, 200; 50, 250; 150, 150; 150, 100
18.	125 250 125	87.5, 187.5; 87.5, 287.5; 212.5, 162.5; 212.5, 62.5
19.	250 125 125	150, 150; 150, 250; 250, 150; 250, 50
20.	150 150 200	50, 200; 50, 250; 200, 200; 200, 50
21.	150 200 150	75, 175; 75, 225; 83.3, 283.3; 150, 250; 225, 175; 225, 25
22.	200 150 150	100, 150; 100, 300; 250, 150; 250, 0
23.	100 275 125	87.5, 187, 5; 87.5, 237.5; 200, 125; 200, 75
24.	125 275 100	100, 175; 100, 250; 212.5, 137.5; 212.5, 62.5
25.	233.33 233.33 33.33	233.33, 133.33; 233.33, 166.67
26.	33.33 233.33 233.33	NO PROPER SCALES
27.	166.7 166.7 166.7	66.67, 183.33; 66.67, 266.67; 88.89, 288.89; 133.33, 266.67; 233.33, 166.67; 233.33, 16.67

6 Scales, modes, and systems

The formation of heptatonic scales from tetrachords was mentioned briefly in chapters 1 and 5. In the present chapter, scale construction will be examined at greater length—in particular, the formation of non-traditional and non-heptatonic scales from tetrachordal modules. Before introducing this new material, however, a brief review of the salient features of the Greek theoretical system is necessary as an introduction to scale construction.

The hierarchy of scalar formations

The ancient Greek theorists recognized a hierarchy of increasingly large scalar formations: tetrachord, pentachord, hexachord, heptachord, octachord, and system. The canonical forms of each of these scalar formations may be seen in 6-1. The smaller formations were finally absorbed into the Perfect Immutable System which with its fifteen pitch keys or tonoi was the highest structural level of the Greek theoretical doctrine. As the tetrachordal level has been introduced in earlier chapters, the discussion will focus on the pentachord and larger structures.

The pentachord

Pentachords may be considered as tetrachords with disjunctive tones added at either extremity. They divide the perfect fifth into four subintervals and occur in several forms in the various modes of heptatonic scales. The two forms of greatest theoretical importance are described in 6-1. While of relatively minor musical prominence, the pentachord has considerable pedagogical value in explaining how certain tunings and scales may have arisen.

For example, Archytas's complex septimal tuning system can be best understood by considering not just the three species of tetrachord, but the pentachords formed with the note a whole tone below. These would be the note hyperhypate for the meson tetrachord and mese for the diezeugmenon (Winnington-Ingram 1932; Erickson 1965). By the use of the harmonic mean between hyperhypate (8/9) and mese (4/3), Archytas defined his enharmonic lichanos as 16/15. His tuning for the note parhypate (28/27) in all three genera was placed as the arithmetic mean between the 8/9 and 32/27, the diatonic lichanos. This construction may be seen in 6-2.

The notes D F G and A form the harmonic series 6:7:8:9 and the notes D G𝄫 A a minor triad, 10:12:15. The 7/6 which the hyperhypate (D) makes with parhypate (F) is found in all three of his genera and is duplicated a fifth higher between mese (A) and trite (C). This interval was very important in Greek theory and had its own name, ekbole (Steinmayer 1985). It occurs in the Dorian harmonia shown in 6-4 and in the fragments of surviving Greek music.

As this interval has the value of 7/6 only in Archytas's tunings and those others of the 7/6 pentachordal family (chapter 4), it is interesting to consider analogous pentachords with the 28/27 replaced by other intervals. 6-2 also depicts such a system, employing a more Aristoxenian 1/4-tone interval, 40/39, which was used by the theorists Eratosthenes, Avicenna, and Barbour in their genera (See the Main Catalog and 4-3). This system has a number of interesting harmonic and melodic intervals and could be played very well in 24-tone equal temperament.

Miscellaneous pentachordal structures

According to Xenakis, chains of conjunct tetrachords and pentachords (trochos) are used in the liturgical music of the Greek Orthodox church

6-1. The hierarchy of scalar formations. The tetrachord may be any of the those listed in chapter 9. The interval of equivalence is the 4/3. The two canonical forms of the pentachord are given. Other forms occur in the various modes of heptatonic scales of different genera and may have the 9/8 interpolated between the tetrachordal intervals. With the addition of the octave 2/1, the heptachord becomes the Mixolydian mode of the complete heptatonic or octachordal scale. If the 8/9 is added below the 1/1 the scale becomes the Hypodorian mode transposed downwards by a whole tone (9/8). The next highest structural level is that of a system which contains all the lower ones. The octachord is the heptatonic Dorian mode.

FORM		NOTES
TETRACHORD		1/1 *a* *b* 4/3
PENTACHORD	1:	1/1 *a* *b* 4/3 3/2
	2:	8/9 1/1 *a* *b* 4/3
HEXACHORD	1:	1/1 *a* *b* 4/3 3/2 3*b*/2
	2:	1/1 *a* *b* 4/3 3/2 3*a*/2
HEPTACHORD		1/1 *a* *b* 4/3 4*a*/3 4*b*/3 16/9
OCTACHORD		1/1 *a* *b* 4/3 3/2 3*a*/2 3*b*/2 2/1

(Xenakis 1971, and chapters 2 and 5). These chains exhibit cyclic permutation of their constituent intervals. Most importantly, they are examples of those rare musical systems in which the octave is not the modulus or interval of equivalence.

6-2. *Pentachordal systems.*

ARCHYTAS'S SYSTEM

D	E	F	G♭♭	G♭	G	A
8/9	1/1	28/27	16/15	9/8	32/27	4/3
6/5			5/4			
7/6		9/7				
7/6		8/7				

40/39 SYSTEM

D	E	F	G♭♭	G♭	G	A
8/9	1/1	40/39	16/15	10/9	32/27	4/3
6/5			5/4			
15/13		13/10				
5/4				6/5		
15/13		52/45				

Additionally, more traditional heptatonic modes (echoi), some of which appear to have genetic continuity with classic Greek theory, if not practice, are employed. These may be analyzed either as composed of two tetrachords or as as combinations of tetrachord and pentachords. A number of tetrachords from these modes are listed in the Catalogs.

Some irregular species of Greek and Islamic origin are also listed in chapter 8 along with Kathleen Schlesinger's harmoniai to which they bear some resemblance. These divide the fourth into four parts and the fifth into five. The Greek forms are merely didactic patterns taken from Aristoxenos and interpreted by Kathleen Schlesinger as support for her theories, while the Islamic scales were apparently modes used in actual music. 8- or 9-tone pseudo-tetrachordal octave scales may be formed by combining these with appropriate fifths or fourths.

The hexachord, heptachord, and gapped scales

The hexachord and heptachord generally appear as transitional forms between the single tetrachord and the complete heptatonic scale or octachord. The hexachord appears as a stage in the evolution of the enharmonic genus from a semitonal pentatonic scale similar to that of the modern Japanese koto to the complete heptatonic octave. This 5-note scale is often called the enharmonic of Olympos (6-3) after the legendary musician who was credited with its discovery by Plutarch (Perrett 1926). This and other pentatonic scales may be construed as two trichords combined with a whole tone to complete the octave. The two intervals of the trichord may be a semitone with a major third, a whole tone with a minor third, or any other combination of two intervals whose sum equals a perfect fourth.

At some point the semitone in the lower trichord was divided into two dieses. This produced the spondeion or libation mode which consisted of a lower enharmonic tetrachord combined by disjunction with an upper trichord consisting of a semitone and a major third (6-3). This hexachord or hexatonic scale evolved into the spondeiakos or spondeiazon tropos. Eventually the semitone in the upper trichord was also split and a hep-

tatonic scale in the enharmonic genus resulted. This transformation may have been completed about the time of Plato, who writes as if he distrusted these innovations. In later times, the ancient pentatonic and hexatonic melodic patterns were retained in compositions for voice and accompaniment (Winnington-Ingram 1936).

In principle, a hexachord can be obtained from a heptatonic scale in four ways by omitting one tone in either tetrachord. 6-3 lists the versions found in the literature. In these cases, the omitted note is the sixth degree, though the second version which lacks the seventh instead is a plausible interpretation in some cases. Schlesinger's version is based on her theories which are described in detail in chapter 8.

Some controversy, however, exists in the literature about the tuning of these early gapped or transilient scales. The arguments over the relative merits of enharmonic or diatonic tunings were discussed by Winnington-Ingram (1928) whose scales and notation are reproduced in 6-3. Notable are his and Mountford's undecimal or 11-limit tunings for the pentatonic forms. Winnington-Ingram's undecimal neutral third pentatonic could be the progenitor of the hemiolic chromatic genus (75 + 75 + 350 cents) and diatonics similar to the equable diatonic such as 150 + 150 + 200 cents. Henderson (1942) has also offered two quite different non-standard interpretations of the enharmonic pentatonic based on etymological considerations.

The hypothetical diatonic versions of these scales according to the suggestions of several scholars are listed in this table as well. Weil and Reinach provide a conventional diatonic form (Winnington-Ingram 1928). The version of Greif appears to be derived from the Lesser Perfect or Conjunct System with the addition of a tone below the tonic as seen in the Dorian harmonia of 6-4 (ibid.). It should be compared with the ancient non-octaval heptachord which may also be formally derived from the conjunct system (6-1).

The medieval diatonic hexachord of Guido D'Arezzo, c d e f g a c', may be included with these scales too, although it is much later in time. In just intonation, it is usually considered to have the ratios 1/1 9/8 5/4 4/3 3/2 5/3, derived from the Lydian mode of Ptolemy's syntonic diatonic instead of the Pythagorean 1/1 9/8 81/64 4/3 3/2 27/16. In the septimal diatonic tuning of Archytas it would have the ratios 1/1 8/7 9/7 4/3 32/21 12/7.

6-3. *Gapped or irregular scales. The notation used here reproduces that of the references. The plus sign indicates a tone 1/4-tone higher than normal. Unless otherwise noted, no particular tuning is assumed, but either Pythagorean or Archytas's supplemented as required with undecimal ratios would be appropriate historically.*

Pentatonic forms

ENHARMONIC OF OLYMPOS
e f a b c (e')

SPONDEION (WINNINGTON–INGRAM 1928)
e f a b c+ *or* e f+ a b c+
1/1 12/11 4/3 3/2 18/11 (2/1)

SPONDEION (HENDERSON 1942)
f a b d# e+ *or* e e+ f a b

SPONDEION (MOUNTFORD 1923)
1/1 28/27 4/3 3/2 18/11 (2/1)

Hexatonic forms

SPONDEIAKOS *or* SPONDEIAZON TROPOS
(WINNINGTON–INGRAM 1928)
e e+ f a b c
with b+ d' *&* c' *in the accompaniment*

DIATONIC OF WEIL & REINACH
(WINNINGTON–INGRAM 1928)
e f g a b d
with b♭ c & e' *in the accompaniment*

GAPPED SCALE OF TERPANDER & NICOMACHOS
(HELMHOLTZ 1877, 266)
e f g a b d (e')

DIATONIC OF GREIF
(WINNINGTON–INGRAM 1928)
d e f a b♭ c# (d')

SCHLESINGER (1939, 183)
1/1 11/10 11/9 11/8 11/7 1/6 (2/1)

Heptatonic form

CONJUNCT HEPTACHORD
c f g a b♭ c d

6-4. *The oldest harmoniai in three genera.*

Dorian

ENHARMONIC d e f- g𝄫 a b c- d'𝄫 e'
CHROMATIC d e f g♭ a b c d'♭ e'
DIATONIC d e f g a b c d' e'

Phrygian

ENHARMONIC d e f- g𝄫 a b c- d'𝄫 d'
CHROMATIC d e f g♭ a b c d'♭ d'
DIATONIC d e f g a b c d'

Lydian

ENHARMONIC f- g𝄫 a b c- d'𝄫 e' f-'
CHROMATIC f g♭ a b c d'♭ e' f'
DIATONIC f g a b c d' e' f'

Mixolydian

ENHARMONIC B c- d𝄫 d e f- g𝄫 b
CHROMATIC B c d♭ d e f g♭ b
DIATONIC B c d e f (g) (a) b

Syntonolydian

ENHARMONIC B C- d𝄫 e g
CHROMATIC B C d♭ e g
DIATONIC c d e f g
2ND DIATONIC B C d e g

Ionian (Iastian)

ENHARMONIC B C- d𝄫 e g a
CHROMATIC B C d♭ e g a
DIATONIC c e f g a
2ND DIATONIC B C d e g a

The octachord or complete heptatonic scale

The union of a tetrachord and a pentachord creates an octachord or complete heptatonic scale. There is evidence, however, that initially two diatonic tetrachords were combined by conjunction, with a shared note between them, to form a 7-note scale less than an octave in span (6-1). The later addition of a whole tone at the top, bottom, or middle separating the two tetrachords, completed the octave gamut. Traces of this early heptachord may be seen in the construction of the Lesser Perfect System and in the irregular scales of 6-3 and 6-4.

Similarly, two enharmonic tetrachords were joined by disjunction with the 9/8 tone between them to create the Dorian harmonia to which a lower tone was added (6-4). An alternative genesis would connect two pentachords whose extra tones were at their bases to produce the 9-tone Dorian harmonia to which other tones might accrete. By analogy, both the enharmonic and diatonic proto-scales converged to the same multi-octave structures later called by the name of system. In the fifth century BCE the wide ditone or major third of the enharmonic genus was gradually narrowed to a minor or subminor third by a process termed "sweetening." Eventually, this process resulted in the chromatic genus which was raised to the same status as the diatonic and enharmonic genera.

The Greater and Lesser Perfect Systems

However the early evolution of the Greek musical system actually occurred, the result came to be schematized as the Perfect Immutable System. Its construction was as follows: two identical tetrachords of any genus and a disjunctive tone (9/8) formed a central heptatonic scale which became the core of the system. Another identical tetrachord was then added by conjunction at both ends of the scale and disjunctive tone was patched on at the bottom of the whole array. A fifth tetrachord, synemmenon, was inserted conjunctly into the middle of the system to recall the ancient heptachord and to facilitate commonly occurring modulations at the fourth. This supernumerary tetrachord was also a useful pedagogical device to illustrate unusual intervals (Erickson 1965; Steinmayer 1985).

The final results consisted of sets of five tetrachords linked by conjunction and disjunction into arrays of fifteen notes spanning two octaves. These systems, in turn, could be transposed into numerous pitch keys or tonoi, at intervals roughly a semitone apart according to the later authors.

The subset of four alternately conjunct and disjunct tetrachords (hypaton, meson, diezeugmenon, and hyperbolaion) was termed the greater perfect (or complete) system (σψστεμα τελειον μειζον). The three conjunct tetrachords (hypaton, meson, and synemmenon), was called the Lesser Perfect (or Complete) System (σψστημα τελειον ελαττον or ελασσον). Their union was called variously the Changeless System or the Perfect Immutable System (συστημα τελειον αμεταβολον) by different authors.

The Perfect Immutable System

By the fourth century BCE, the Greek theorists had analyzed the scales or harmoniai of their music into sections of this theoretical two octave gamut. This 15-note span is conventionally transcribed into our notation as lying between A and a'. The Perfect Immutable System could be tuned to each of the three genera, and while in theory all five of the tetrachords must be the same, in practice mixed tetrachords and considerable chromaticism occurred. Not only was the diatonic lichanos meson (D in the Dorian or E mode) added, but other extrascalar notes led to successions of more than two semitones (Winnington-Ingram 1936).

6-5 depicts the Perfect Immutable System in its theoretical form and in its two most historically important intonations.

The fixed notes (hestotes) of the Perfect Immutable System were proslambanomenos, hypate hypaton, hypate meson, mese, paramese, nete diezeugmenon, nete hyperbolaion, and nete synemmon. The moveable tones (κινουμενοι) were the parhypatai, the lichanoi, the tritai, and the paranetai of each genus.

Lichanos hypaton, also called hyperhypate, a diatonic note a whole tone (9/8 in Archytas's and most other just tunings) below the tonic, was added to the Dorian octave species in the chromatic and enharmonic genera in the harmoniai of Aristides Quintilianus, certain planetary scales, and the Euripides fragment (ibid.).

Erickson (1965) and Vogel (1963, 1975) have shown that a number of interesting tetrachords occur in the region where the synemmenon tetrachord overlaps with the diezeugmenon tetrachord in Archytas's system. These include the later and historically important 16/15 · 9/8 · 10/9 (Ptolemy's syntonic diatonic), 16/15 · 10/9 · 9/8 (Didymos's diatonic), the three permutations of the Pythagorean diatonic, 256/243 · 9/8 · 9/8, (90 + 204 + 204 cents), the Pythagorean chromatic 32/27 · 2187/2048 · 256/243 (294 +

114 + 90 cents), and Avicenna's chromatic 7/6 · 36/35 · 10/9 (267 + 49 + 182 cents). Some unusual divisions such as 28/27 · 81/70 · 10/9 (63 + 253 + 182 cents), 28/27 · 2187/1792 · 256/243 (63 + 345 + 90 cents), 16/15 · 35/32 · 8/7 (112 + 155 + 231 cents), 16/15 · 1215/1024 · 256/243 (112 + 296 + 90 cents), 7/6 · 81/80 · 9/8 (267 + 22 + 204 cents), 32/27 · 81/80 · 10/9 (294 + 22 + 182 cents), 28/27 · 64/63 · 81/64 (63 + 22 + 408 cents), 6/5 · 135/128 · 256/243 (316 + 92 + 90 cents), and 256/243 · 81/80 · 5/4 (90 + 22 + 386 cents) are also found here. Notable are the intervals of 253 cents, another possible tuning for the ekbole, the neutral third of 345 cents, the three-quarter tone 35/32 (155 cents), and the minor whole tone 10/9.

The alternate tunings 16/15 and 28/27 for the first interval of the synemmenon tetrachord may have been used in order to obtain the spondeiasmos, an interval of three dieses approximating 150 cents, mentioned by Bacchios (Steinmayer 1985; Winnington-Ingram 1932). These intervals would measure 35/32 (155 cents) as the difference between 14/9 and 64/45, or 243/224 (141 cents) as the difference between 112/81 and 3/2. The in-

6-5. The Perfect Immutable System in the diatonic, chromatic, and enharmonic genera, tuned according to Archytas's and Pythagorean tuning. The transcription is in the natural key to avoid accidentals and the mistaken late shift of emphasis from Dorian to Hypolydian (Henderson 1957). The - and ♭̶ indicate that these are different pitches in the enharmonic genus. Erickson (1965) proposes 64/45 as an alternative tuning for trite synemmenon.

	TRANSCRIPTION			ARCHYTAS			PYTHAGOREAN		
	DIA.	CHR.	ENH.	DIA.	CHR.	ENH.	DIA.	CHR.	ENH.
PROSLAMBANOMENOS	A	A	A	2/3	2/3	2/3	2/3	2/3	2/3
HYPATE HYPATON	B	B	B	3/4	3/4	3/4	3/4	3/4	3/4
PARHYPATE HYPATON	C	C	C-	7/9	7/9	7/9	64/81	64/81	384/499
LICHANOS HYPATON	D	D♭	D♭̶	8/9	27/32	4/5	8/9	27/32	64/81
HYPATE MESON	E	E	E	1/1	1/1	1/1	1/1	1/1	1/1
PARHYPATE MESON	F	F	F-	28/27	28/27	28/27	256/243	256/243	512/499
LICHANOS MESON	G	G♭	G♭̶	32/27	9/8	16/15	32/27	9/8	256/243
MESE	a	a	a	4/3	4/3	4/3	4/3	4/3	4/3
PARAMESE	b	b	b	3/2	3/2	3/2	3/2	3/2	3/2
TRITE DIEZEUGMENON	c	c	c-	14/9	14/9	14/9	128/81	128/81	768/499
PARANETE DIEZEUGMENON	d	d♭	d♭̶	16/9	27/16	8/5	16/9	27/16	128/81
NETE DIEZEUGMENON	e	e	e	2/1	2/1	2/1	2/1	2/1	2/1
TRITE HYPERBOLAION	f	f	f-	56/27	56/27	56/27	512/243	512/243	1024/499
PARANETE HYPERBOLAION	g	g♭	g♭̶	64/27	9/4	32/15	64/27	9/4	512/243
NETE HYPERBOLAION	a'	a'	a'	8/3	8/3	8/3	8/3	8/3	8/3
TRITE SYNEMMENON (28/27)	b♭	b♭	b♭-	112/81	112/81	112/81	1024/729	1024/729	2048/1497
PARANETE SYNEMMENON	c	c♭	c♭̶	128/81	3/2	64/45	128/81	3/2	1024/729
NETE SYNEMMENON	d	d	D	16/9	16/9	16/9	16/9	16/9	16/9

3/2. The interval of three dieses also appears in Archytas's chromatic as the difference between the 28/27 and the 9/8. In many cases the scales containing these tetrachords would be mixed, but deliberately mixed scales were not unknown. 6-6 lists some varieties of mixed scales recorded by Ptolemy in the second century CE.

The scales actually employed in Greek music are a matter of some confusion because of the paucity of extant musical examples and the variety of theoretical works from different traditions written over a period of several centuries (fourth century BCE to fourth century CE). In the theoretical treatises, the seven octave species or circular permutations of the basic heptatonic scale are singled out and given names derived from early tribal groups. These scales are notated in all three genera in 6-7. Their intervals and notes are in shown in ratios for both Archytas's and Pythagorean tuning in 6-8 and 6-9. 6-10 gives the diatonic form in Ptolemy's syntonic diatonic (16/15 · 9/8 · 10/9), and 6-11 gives the retrograde of this genus (10/9 · 9/8 · 16/15). The Lydian mode in the former tuning is the standard just intonation of the major scale, and the latter is that of the natural minor mode (see chapter 7).

For the Pythagorean tuning of the enharmonic, I have used Boethius's much later arithmetic division of the pyknon, as the actual tuning prior to Archytas is not known. Since the division of the semitone in both tetra-

1. STEREA, A LYRA TUNING: TONIC DIATONIC
1/1 28/27 32/27 4/3 3/2 14/9 16/9 2/1

2. MALAKA, A LYRA TUNING: SOFT OR INTENSE CHROMATIC AND TONIC DIATONIC
A. 1/1 28/27 10/9 4/3 3/2 14/9 16/9 2/1
B. 1/1 22/21 8/7 4/3 3/2 14/9 16/9 2/1

3. METABOLIKA, ANOTHER LYRA TUNING: SOFT DIATONIC AND TONIC DIATONIC
1/1 21/20 7/6 4/3 3/2 14/9 16/9 2/1

4. IASTI-AIOLIKA, A KITHARA TUNING: TONIC DIATONIC AND DITONIC DIATONIC
1/1 28/27 32/27 4/3 3/2 27/16 16/9 2/1

5. IASTIA OR LYDIA, KITHARA TUNINGS: INTENSE DIATONIC AND TONIC DIATONIC
1/1 28/27 32/27 4/3 3/2 8/5 9/5 2/1

6. A MEDIEVAL ISLAMIC SCALE OF ZALZAL FOR COMPARISON
1/1 9/8 81/64 4/3 40/27 130/81 16/9 2/1

6-6. *Scales in common use according to Ptolemy. In the text, the names of the tunings are always given in plural form. (1), not the ditonic or Pythagorean, appears to have been the standard diatonic. On the kithara, in the Hypodorian mode it was called tritai; in the Phrygian, hypertropa. (2a) is given in two forms in different places in the* Harmonics; *the intense chromatic (1:84), where it is mistranslated as "diatonic chromatic," and the soft chromatic (2:208). The tables (2:178) use the intense chromatic; the soft chromatic fits the sense of the name better. On the kithara, (2b) in the Hypodorian mode is called tropoi or tropikoi. In the Dorian mode on the kithara, (3) is called parypatai. (4) is in the Hypophrygian mode. (5), in the Dorian mode, is given variously as either pure tonic diatonic or a mixture of tonic diatonic and intense and is also referred to as metabolika. (6) is from Avicenna (D'Erlanger 1935, 2:239), who sometimes approximated complex ratios like 72/65 with superparticulars of similar magnitude such as 22/21, but the exact ratio is clear from the context.*

chords was completed only near end of the fourth century BCE, the division may not have been standardized and was most likely done by ear during the course of the melody (Winnington-Ingram 1928), in which case the approximate equality of the dieses in Boethius's tuning probably captures the flavor of the scale adequately. Euler's eighteenth-century tuning (Euler [1739] 1960, and Catalog number 79) is similar and considerably simpler. An impractical, if purely Pythagorean, solution (number 81) as well as some other approximations are given in the Main Catalog.

Although these scales are analogous to the "white key" modes, the latter are named out of order due to a misunderstanding in early medieval times.

TONIC	NAME	MESE
	Diatonic	
(A	HYPERMIXOLYDIAN, HYPERPHRYGIAN, LOCRIAN	D)
B	MIXOLYDIAN, HYPERDORIAN	E
C	LYDIAN	F
D	PHRYGIAN	G
E	DORIAN	a
F	HYPOLYDIAN	b
G	HYPOPHRYGIAN, IONIAN	c
a	HYPODORIAN, AEOLIAN	d
	Chromatic	
(A	HYPERMIXOLYDIAN, HYPERPHRYGIAN, LOCRIAN	D♭)
B	MIXOLYDIAN, HYPERDORIAN	E
C	LYDIAN	F
D♭	PHRYGIAN	G♭
E	DORIAN	a
F	HYPOLYDIAN	b
G♭	HYPOPHRYGIAN, IONIAN	c
a	HYPODORIAN, AEOLIAN	DB
	Enharmonic	
(A	HYPERMIXOLYDIAN, HYPERPHRYGIAN, LOCRIAN	D𝄫)
B	MIXOLYDIAN, HYPERDORIAN	E
C-	LYDIAN	F-
D𝄫	PHRYGIAN	G𝄫
E	DORIAN	a
F-	HYPOLYDIAN	b
G𝄫	HYPOPHRYGIAN, IONIAN	c-
a	HYPODORIAN, AEOLIAN	d𝄫

6-7. The octave species in all three genera. The traditional names are given first and alternate ones subsequently. The Hypermixolydian was denounced by Ptolemy as otiose and by the city of Argos as illegal (Winnington-Ingram 1936). This transcription uses the natural key for clarity. Late theorists mistakenly built the system and notation about the F mode (Hypolydian) rather than the correct E mode (Dorian) (Henderson 1957). Although the Dorian, Phrygian, and Lydian modes have distinctive tetrachordal forms, these forms were never named after their parent modes by any of the Greek theorists. In the chromatic and enharmonic genera the tonics of the species are transformed. An alternative nomenclature for the enharmonic tetrachord is E E+ F A. The mese kata thesin is four scale degrees above the tonic with which it usually makes an interval of a perfect fourth.

Diatonic (28/27 · 8/7 · 9/8)

MIXOLYDIAN (B – b)

1/1 28/27 32/27 4/3 112/81 128/81 16/9 2/1
28/27 · 8/7 · 9/8 · 28/27 · 8/7 · 9/8 · 9/8

LYDIAN (C – c)

1/1 8/7 9/7 4/3 32/21 12/7 27/14 2/1
8/7 · 9/8 · 28/27 · 8/7 · 9/8 · 9/8 · 28/27

PHRYGIAN (D – d)

1/1 9/8 7/6 4/3 3/2 27/16 7/4 2/1
9/8 · 28/27 · 8/7 · 9/8 · 9/8 · 28/27 · 8/7

DORIAN (E – e)

1/1 28/27 32/27 4/3 3/2 4/9 16/9 2/1
28/27 · 8/7 · 9/8 · 9/8 · 28/27 · 8/7 · 9/8

HYPOLYDIAN (F – f)

1/1 8/7 9/7 81/56 3/2 12/7 27/14 2/1
8/7 · 9/8 · 9/8 · 28/27 · 8/7 · 9/8 · 28/27

HYPOPHRYGIAN (G – g)

1/1 9/8 81/64 21/16 3/2 27/16 7/4 2/1
9/8 · 9/8 · 28/27 · 8/7 · 9/8 · 28/27 · 8/7

HYPODORIAN (A – a)

1/1 9/8 7/6 4/3 3/2 14/9 16/9 2/1
9/8 · 28/27 · 8/7 · 9/8 · 28/27 · 8/7 · 9/8

Chromatic (28/27 · 243/224 · 32/27)

MIXOLYDIAN (B – b)

1/1 28/27 9/8 4/3 112/81 3/2 16/9 2/1
28/27 · 243/224 · 32/27 · 28/27 · 243/224 · 32/27 · 9/8

LYDIAN (C – c)

1/1 243/224 9/7 4/3 81/56 12/7 27/14 2/1
243/224 · 32/27 · 28/27 · 243/224 · 32/27 · 9/8 · 28/27

PHRYGIAN (D♭ – d♭)

1/1 32/27 896/729 4/3 128/81 16/9 448/243 2/1
32/27 · 28/27 · 243/224 · 32/27 · 9/8 · 28/27 · 243/224

DORIAN (E – e)

1/1 28/27 9/8 4/3 3/2 14/9 27/16 2/1
28/27 · 243/224 · 32/27 · 9/8 · 28/27 · 243/224 · 32/27

HYPOLYDIAN (F – f)

1/1 243/224 9/7 81/56 3/2 729/448 27/14 2/1
243/224 · 32/27 · 9/8 · 28/27 · 243/224 · 32/27 · 28/27

HYPOPHRYGIAN (G♭ – g♭)

1/1 32/27 4/3 112/81 3/2 16/9 448/243 2/1
32/27 · 9/8 · 28/27 · 243/224 · 32/27 · 28/27 · 243/224

HYPODORIAN

1/1 9/8 7/6 81/64 3/2 14/9 27/16 2/1
9/8 · 28/27 · 243/224 · 32/27 · 28/27 · 243/224 · 32/27

Enharmonic (28/27 · 36/35 · 5/4)

MIXOLYDIAN (B – b)

1/1 28/27 16/15 4/3 112/81 64/45 16/9 2/1
28/27 · 36/35 · 5/4 · 28/27 · 36/35 · 5/4 · 9/8

LYDIAN (C- – c-)

1/1 36/35 9/7 4/3 48/35 12/7 27/14 2/1
36/35 · 5/4 · 28/27 · 36/35 · 5/4 · 9/8 · 28/27

PHRYGIAN (D𝄫 – d𝄫)

1/1 5/4 35/27 4/3 5/3 15/8 35/18 2/1
5/4 · 28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35

DORIAN (E – e)

1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1
28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35 · 5/4

HYPOLYDIAN (F- – f-)

1/1 36/35 9/7 81/56 3/2 54/35 27/14 2/1
36/35 · 5/4 · 9/8 · 28/27 · 36/35 · 5/4 · 28/27

HYPOPHRYGIAN (G𝄫 – g𝄫)

1/1 5/4 45/32 35/24 3/2 15/8 35/18 2/1
5/4 · 9/8 · 28/27 · 36/35 · 5/4 · 28/27 · 36/35

HYPODORIAN (A – a)

1/1 9/8 7/6 6/5 3/2 14/9 8/5 2/1
9/8 · 28/27 · 36/35 · 5/4 · 28/27 · 36/35 · 5/4

6-8. *The intervals of the octave species in all three genera in Archytas's tuning.*

Diatonic (256/243 · 9/8 · 9/8)

MIXOLYDIAN (B – b)

1/1 256/243 32/27 4/3 1024/729 128/81 16/9 2/1

256/243 · 9/8 · 9/8 · 256/243 · 9/8 · 9/8 · 9/8

LYDIAN (C – c)

1/1 9/8 81/64 4/3 3/2 27/16 243/128 2/1

9/8 · 9/8 · 256/243 · 9/8 · 9/8 · 9/8 · 256/243

PHRYGIAN (D – d)

1/1 9/8 32/27 4/3 3/2 27/16 16/9 2/1

9/8 · 256/243 · 9/8 · 9/8 · 9/8 · 256/243 · 9/8

DORIAN (E – e)

1/1 256/243 32/27 4/3 3/2 128/81 16/9 2/1

256/243 · 9/8 · 9/8 · 9/8 · 256/243 · 9/8 · 9/8

HYPOLYDIAN (F – f)

1/1 9/8 81/64 729/512 3/2 27/16 243/128 2/1

9/8 · 9/8 · 9/8 · 256/243 · 9/8 · 9/8 · 256/243

HYPOPHRYGIAN (G – g)

1/1 9/8 81/64 4/3 3/2 27/16 16/9 2/1

9/8 · 9/8 · 256/243 · 9/8 · 9/8 · 256/243 · 9/8

HYPODORIAN (A – a)

1/1 9/8 32/27 4/3 3/2 128/81 16/9 2/1

9/8 · 256/243 · 9/8 · 9/8 · 256/243 · 9/8 · 9/8

Chromatic (256 · 2187/2028 · 32/27)

MIXOLYDIAN (B – b)

1/1 256/243 9/8 4/3 1024/729 3/2 16/9 2/1

256/243 · 2187/2048 · 32/27 · 256/243 · 2187/2048 · 32/27 · 9/8

LYDIAN (C – c)

1/1 2187/2048 81/64 4/3 729/512 27/16 243/128 2/1

2187/2048 · 32/27 · 256/243 · 2187/2048 · 32/27 · 9/8 · 256/243

PHRYGIAN (D♭ – d♭)

1/1 32/27 8192/6561 4/3 128/81 16/9 4096/2187 2/1

32/27 · 256/243 · 2187/2048 · 32/27 · 9/8 · 256/243 · 2187/2048

DORIAN (E – e)

1/1 256/243 9/8 4/3 3/2 128/81 27/16 2/1

256/243 · 2187/2048 · 32/27 · 9/8 · 256/243 · 2187/2048 · 32/27

HYPOLYDIAN (F – f)

1/1 2187/2048 81/64 729/512 3/2 6561/4096 243/128 2/1

2187/2048 · 32/27 · 9/8 · 256/243 · 2187/2048 · 32/27 · 256/243

HYPOPHRYGIAN (G♭ – g♭)

1/1 32/27 4/3 729/512 3/2 16/9 4096/2187 2/1

32/27 · 9/8 · 256/243 · 2187/2048 · 32/27 · 256/243 · 2187/2048

HYPODORIAN (A – a)

1/1 9/8 32/27 81/64 3/2 128/81 27/16 2/1

9/8 · 256/243 · 2187/2048 · 32/27 · 256/243 · 2187/2048 · 32/27

Enharmonic (512/499 · 499/486 · 81/64)

MIXOLYDIAN (B – b)

1/1 512/499 256/243 4/3 2048/1497 1024/729 16/9 2/1

512/499 · 499/486 · 81/64 · 512/499 · 499/486 · 81/64 · 9/8

LYDIAN (C- – c-)

1/1 499/486 499/384 4/3 998/729 499/288 499/256 2/1

499/486 · 81/64 · 512/499 · 499/486 · 81/64 · 9/8 · 512/499

PHRYGIAN (D𝄫 – d𝄫)

1/1 81/64 648/499 4/3 27/16 243/128 972/499 2/1

81/64 · 512/499 · 499/486 · 81/64 · 9/8 · 512/499 · 499/486

DORIAN (E – e)

1/1 512/499 256/243 4/3 3/2 768/499 128/81 2/1

512/499 · 499/486 · 81/64 · 9/8 · 512/499 · 499/486 · 81/64

HYPOLYDIAN (F- – f-)

1/1 499/486 499/384 1497/1024 3/2 499/324 499/256 2/1

499/486 · 81/64 · 9/8 · 512/499 · 499/486 · 81/64 · 512/499

HYPOPHRYGIAN (G𝄫 – g𝄫)

1/1 81/64 729/512 729/499 3/2 243/128 972/499 2/1

81/64 · 9/8 · 512/499 · 499/486 · 81/64 · 512/499 · 499/486

HYPODORIAN (A – a)

1/1 9/8 576/499 32/27 3/2 768/499 128/81 2/1

9/8 · 512/499 · 499/486 · 81/64 · 512/499 · 499/486 · 81/64

6-9. *The intervals of the octave species in Pythagorean tuning. The tuning of the pre-Archytas enharmonic is not known, but at first it had undivided semitones, obtaining the pyknon later. Boethius's tuning is used here.*

6-10. *The intervals of the octave species of Ptolemy's intense diatonic genus. See figures 6-3 and 6-6 for names of notes. The diatonic tetrachord is 16/15 · 9/8 · 10/9. The Lydian mode in this tuning is the major mode in just intonation. The Hypodorian or A mode is not the minor mode as the fourth degree is 27/20 instead of 4/3.*

6-11. *The intervals of the octave species of the Ptolemy's intense diatonic genus, reversed. The diatonic tetrachord is 10/9 · 9/8 · 16/15. The Lydian or C mode in this tuning is the minor mode in just intonation. The Dorian or E mode is not the major mode as the second degree is 10/9 instead of 9/8. This scale transposed to C is John Redfield's tuning for the major scale (Redfield 1928).*

Although they are conventionally presented as sections of the two octave gamut, they were actually retunings of the central octave so that the sequences of intervals corresponding to the cyclic modes fell on the notes of the Perfect Immutable System (hypate meson to nete diezeugemenon, e to e'). These abstract sequences of intervals are shown in 6-12. Thus, in the Dorian tonos, the interval sequence of the Dorian mode filled the central octave; in the Phrygian, the Phrygian sequence was central and the Dorian, a tone higher. In the Hypolydian tonos, the initial A, proslambanomenos, was raised a semitone, as was its octave, mese, the supposed tonal center of the whole system.

From the original set of seven pitch keys (tonoi), a later set of thirteen or fifteen theoretical keys at more or less arbitrary semitonal intervals developed, irrespective of genus (Crocker 1966; Winnington-Ingram 1936). In Roman times, the theorists moved the entire system up a semitone so

MIXOLYDIAN (B – b)

1/1 16/15 6/5 4/3 64/45 8/5 16/9 2/1

16/15 · 9/8 · 10/9 · 16/15 · 9/8 · 10/9 · 9/8

LYDIAN (C – c)

1/1 9/8 5/4 4/3 3/2 5/3 15/8 2/1

9/8 · 10/9 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15

PHRYGIAN (D – d)

1/1 10/9 32/27 4/3 40/27 5/3 16/9 2/1

10/9 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 9/8

DORIAN (E – e)

1/1 16/15 6/5 4/3 3/2 8/5 9/5 2/1

16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9

HYPOLYDIAN (F – f)

1/1 9/8 5/4 45/32 3/2 27/16 15/8 2/1

9/8 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 16/15

HYPOPHRYGIAN (G – g)

1/1 10/9 5/4 4/3 3/2 5/3 16/9 2/1

10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 16/15 · 9/8

HYPODORIAN (A – a)

1/1 9/8 6/5 27/20 3/2 8/5 9/5 2/1

9/8 · 16/15 · 9/8 · 10/9 · 16/15 · 9/8 · 10/9

MIXOLYDIAN (B – b)

1/1 10/9 5/4 4/3 40/27 5/3 16/9 2/1

10/9 · 9/8 · 16/15 · 10/9 · 9/8 · 16/15 · 9/8

LYDIAN (C – c)

1/1 9/8 6/5 4/3 3/2 8/5 9/5 2/1

9/8 · 16/15 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9

PHRYGIAN (D – d)

1/1 16/15 32/27 4/3 64/45 8/5 16/9 2/1

16/15 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 9/8

DORIAN (E – e)

1/1 10/9 5/4 4/3 3/2 5/3 15/8 2/1

10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15

HYPOLYDIAN (F – f)

1/1 9/8 6/5 27/20 3/2 27/16 9/5 2/1

9/8 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 10/9

HYPOPHRYGIAN (G – g)

1/1 16/15 6/5 4/3 3/2 8/5 16/9 2/1

16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 10/9 · 9/8

HYPODORIAN (A – a)

1/1 9/8 5/4 45/32 3/2 5/3 15/8 2/1

9/8 · 10/9 · 9/8 · 16/15 · 10/9 · 9/8 · 16/15

6-12. *Interval sequences of the octave species of the abstract tetrachord* a · b · c. a · b · c = 4/3 (c = 4/3ab) *in just intonation or* a + b + 500 - a - b *with the disjunctive tone equaling 200 cents in the zero modulo 12 equal temperaments. In the Main Catalog, c is equal to the CI.*

MIXOLYDIAN *a · b · c · a · b · c ·* 9/8	HYPOLYDIAN *b · c ·* 9/8 *· a · b · c · a*
LYDIAN *b · c · a · b · c ·* 9/8 *· a*	HYPOPHRYGIAN *c ·* 9/8 *· a · b · c · a · b*
PHRYGIAN *c · a · b · c ·* 9/8 *· a · b*	HYPODORIAN 9/8 *· a · b · c · a · b · c*
DORIAN *a · b · c ·* 9/8 *· a · b · c*	

that the central octave began on either E or F in modern notation. In this final form, however, the central octave had the interval sequence of the Hypolydian mode rather than the Dorian.

The modal retunings could also be considered as transpositions of the entire Perfect Immutable System. The order of the keys ran in the opposite direction to that of the homonymous octave species and the octave species could be described either by the positions of their interval sequences in relation to the untransposed Dorian or by the relative pitch of the entire Perfect Immutable System. This duality is reflected in the two nomenclatures employed by Ptolemy, the "onomasia kata thesin" (by position) and "onomasia kata dynamin" (by function). The thetic nomenclature in the natural key is used in the tables of this chapter and chapter 8 as it is the same for all tonoi. The dynamic refers all notes to the Dorian tonos for which the thetic and dynamic nomenclatures are identical.

6-13. *Vogel's transcription of the Greek notations. Only the upper octave from mese to nete hyperbolaion is shown. Vogel's German notation has been transcribed into the American form. His notes have been transposed up an octave, and those marked with a bar in the original are given a + here. 512/405 (406 cents) replaces 81/64 (408 cents), in Vogel's tuning. In the upper half of the scale, 2048/1215 replaces 27/16.*

NOTE	RATIO	NOTATION
MESE	1/1	A
TRITE SYNEMMENON	28/27	B♭-
PARANETE SYNEMMENON	16/15 (ENHARMONIC)	B♭+
PARANETE SYNEMMENON, PARAMESE	9/8 (CHROMATIC)	B
TRITE DIEZEUGMENON	7/6	C-
PARANETE SYNEMMENON	32/27 (DIATONIC)	C
PARANETE DIEZEUGMENON	6/5 (ENHARMONIC)	C+
	896/729	D♭-
	512/405 (CHROMATIC)	D♭+
	4/3 (DIATONIC)	D
NETE SYNEMMENON	112/81	E♭-
	64/45	E♭
NETE DIEZEUGMENON	3/2	E
TRITE HYPERBOLAION	14/9	F-
PARANETE HYPERBOLAION	8/5 (ENHARMONIC)	F+
	128/81	F
	3584/2187	G♭-
PARANETE HYPERBOLAION	2048/1215 (CHROMATIC)	G♭
	16/9 (DIATONIC)	G
	448/243	A♭-
	256/135	A♭
NETE HYPERBOLAION	2/1	A

6-14. *Unusual tetrachords in Vogel's transcription.*

RATIOS	CENTS
64/63 · 81/80 · 35/27	27 + 22 + 449
81/80 · 2240/2187 · 9/7	22 + 41 + 435
36/35 · 2240/2187 · 81/84	49 + 41 + 408
36/35 · 256/243 · 315/256	49 + 90 + 359
64/63 · 16/15 · 315/256	27 + 112 + 359
64/63 · 2187/2048 · 896/729	27 + 114 + 357
896/729 · 36/35 · 135/128	357 + 49 + 92
28/27 · 256/243 · 2187/1792	63 + 90 + 345
16/15 · 2240/2187 · 2187/1792	112 + 41 + 345
28/27 · 128/105 · 135/128	63 + 343 + 92
6/5 · 35/32 · 64/63	316 + 155 + 27
6/5 · 2240/2187 · 243/224	316 + 41 + 141
7168/6561 · 36/35 · 1215/1024	153 + 49 + 296
16/15 · 1215/1024 · 256/243	112 + 296 + 90
28/27 · 1024/945 · 1215/1024	63 + 139 + 296
7/6 · 1024/945 · 135/128	267 + 139 + 92
28/27 · 81/70 · 10/9	63 + 253 + 182
81/70 · 2240/2187 · 9/8	253 + 41 + 204
81/70 · 256/243 · 35/32	253 + 90 + 155
135/128 · 7168/6561 · 81/70	92 + 153 + 253
16/15 · 280/243 · 243/224	112 + 245 + 141
36/35 · 9/8 · 280/243	49 + 204 + 245
8/7 · 81/80 · 280/243	231 + 22 + 245
9/8 · 7168/6561 · 243/224	204 + 153 + 141
9/8 · 4096/3645 · 135/128	204 + 202 + 92
35/32 · 1024/945 · 9/8	155 + 139 + 204
4096/3645 · 35/32 · 243/224	202 + 155 + 141

The Greeks named the modes from their keynotes as octave species of the Perfect Immutable System, while the medieval theorists named them in order of their transpositions (Sachs 1943). The two concepts became confused by the time of Boethius. For this reason the names of the ecclesiastical modes are different from those of ancient Greece. In more recent periods, other ecclesiastical nomenclatures were developed.

Greek alphabetic notations

In addition to the thetic and dynamic nomenclatures, which were really tablatures derived from the names of the strings of the kithara or similar instrument, there were two alphabetical cipher notations, the vocal and the instrumental. These were recorded for the each of the tonoi in all three genera by the theorist Alypius. The independent elucidation of Alypius's tables by Bellermann (1847) and Fortlage (1847) have permitted scholars to transcribe the few extant fragments of Greek music into modern notation.

Vogel (1963, 1967) has translated these cipher notations into a tuning system based on Archytas's and Pythagoras's genera (6-4). This set of tones includes a number of unusual tetrachords, most of which occur in several permutations (6-13). Some of these are good approximations to the neo-Aristoxenian types: 50 + 100 + 350 cents, 50 + 150 + 300 cents, 50 + 250 + 200 cents, and 150 + 150 + 200 cents of chapter 4.

The Greek notations, however, were not entirely without ambiguity, and some uncertainly exists over the meaning of certain presumed "enharmonic" equivalences, i.e. two notes of the same pitch written differently. Kathleen Schlesinger developed her somewhat fantastic theories, detailed in chapter 8, in part from deliberations on the apparent anomalies of these notations.

Concise descriptions of the notational systems may be found in Sachs (1943) and Henderson (1957).

The oldest harmoniai or modes

Although the melodic canons laid down by Aristoxenos (330 BCE) stated that the smallest interval the melody could move from the pyknon was a whole tone and that notes four or five positions apart must make either perfect fourths or fifths, both literary evidence and the surviving fragments attest to mixed scales and chromaticism (Winnington-Ingram 1936), as mentioned previously. A late writer, Aristides Quintilianus, gave a list of what he said were the scales approved by Plato in the *Republic*. These scales

are in the enharmonic genus and depart quite strongly from the conventional octave species of 6-7. Since it is known that both diatonic and chromatic scales of the same name existed, it is tempting to try to reconstruct them. 6-4 contains Aristides's enharmonic harmoniai, Henderson's (1942) diatonic versions, and my own chromatic and diatonic forms. The chromatic versions are based on Winnington-Ingram's indication that there is literary evidence for certain chromatic versions (1936). The diatonic harmoniai are from Henderson (1942), except in the cases of the Syntonolydian and Iastian where I have supplied a second diatonic which I feel better preserves the melodic contours. In the enharmonic and chromatic forms of some of the harmoniai, it has been necessary to use both a d and either a d♭ or d♭♭ because of the non-heptatonic nature of these scales. C and F are synonyms for d♭♭ and g♭♭. The appropriate tunings for these scales are those of Archytas (Mountford 1923) and Pythagoras.

These scales are very important evidence for the use of extrascalar tones (diatonic lichanos meson, called hyperhypate) and scalar gaps, which were alluded to by Aristoxenos as an indispensable ingredient in determining the ethos of the mode. Furthermore, one of the fragments, a portion of the first stationary chorus of Euripides's *Orestes*, uses hyperhypate and the enharmonic in such a way as to prove that the middle tone of the pyknon (mesopyknon) was not merely a grace note, but a full member of the scale (Winnington-Ingram 1936).

Ptolemy's mixed scales

Still more remote from the conventional theory are the mixed scales listed by Ptolemy in the *Harmonics*. These scales are ones that he said were in common use by players of the lyra and kithara in Alexandria in the second century CE (6-6). These scales bear some resemblance to modern Islamic modes containing 3/4-tone intervals, as does Ptolemy's equable diatonic, 12/11 · 11/10 · 10/9. They offer important support and evidence for the combination of tetrachords of varying genera and species to generate new musical materials.

Permutation of intervals

Although traditional techniques can generate a wealth of interesting material for musical exploration, the Greek writers suggested only a small fraction of the possibilities inherent in the permutations and combinations of tetrachords. While Aristoxenos mentioned the varying arrangements of

the intervals of the tetrachord in the different octave species, the Islamic theorists, such as Safiyu-d-Din, gave lengthy tables of all the permutational forms of tetrachords with two and three different intervals. However, the construction of 5-, 6-, and 7-tone scales from permuted tetrachords and trichords (gapped tetrachords) has been studied most thoroughly by the composer Lou Harrison (1975). Harrison constructed scales from all the permutations of the tetrachords and trichords and allowed different permutations in the upper and lower parts of the scale.

In chapter 5, the melodic properties of scales constructed of either identical or dissimilar tetrachords, irrespective of permutational order, are analyzed according to the perception theories of David Rothenberg (1969, 1975, 1978; also Chalmers 1975).

Wilson's permutations and modulations

Perhaps the most sophisticated use to date of tetrachordal interval permutation in a generative sense is Ervin Wilson's derivation of certain North Indian thats (raga-scales) and their analogs (Wilson 1986a; 1987). In "The Marwa Permutations" (1986a), Wilson's procedure is to permute the order of the sequential fourths of heptatonic scales constructed from two identical tetrachords. These sequential fourths are computed in the usual manner by starting with the lowest note of one of the modes and counting three melodic steps upwards. The process is continued until the cycle is complete and one is back to the original tone. The resulting seven fourths are the same as the adjacent fourths of the difference matrices of chapter 5, but in a different order. In abstract terms, if the intervals of the tetrachord are $a \cdot b/a \cdot 4/3b$, the scale is $1/1\ a\ b\ 4/3\ 3/2\ 3a/2\ 3b/2$, and 2/1. The sequential fourths from 1/1 are thus $4/3$, $3/2a$, $3a/2b$, $9b/8$, 4/3, 4/3, and 4/3. It is clear that these fourths must be of at least two different sizes even in Pythagorean intonation.

While holding the position of one fourth constant to avoid generating cyclic permutations or modes, pairs of fourths are exchanged to create new sequences of intervals in general not obtainable by the traditional modal operations. Both the choice of the positionally fixed fourth and the arrangement of the tetrachordal intervals affect the spectrum of scales obtainable from a given genus.

6-15 illustrates this process with the enharmonic genus of Archytas. The exchange of the second and third fourths converts the upper tetrachord into

6-15. *Permutations of sequential fourths. See Wilson 1986 for further details. This example begins with the Dorian mode of the standard ascending form for clarity and consistency with other sections of this treatise. The sizes of the fourths range from 6/5 (316 cents) to 35/24 (653 cents). Interval 7 in the original sequence is a fixed fourth. The pair of permuted fourths are in boldface. The last tetrachord is Archytas's diatonic.*

ORIGINAL SCALE

1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1
28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35 · 5/4

	FOURTHS	SIZE
1.	1/1 *to* 4/3	4/3
2.	4/3 *to* 8/5	6/5
3.	8/5 *to* 16/15	4/3
4.	16/15 *to* 14/9	35/24
5.	14/9 *to* 28/27	4/3
6.	28/27 *to* 3/2	81/56
7.	3/2 *to* 2/1	4/3

ORIGINAL SEQUENCE

1	2	3	4	5	6	7
4/3	**6/5**	**4/3**	35/24	4/3	81/56	(4/3)

PERMUTED SEQUENCE

1	3	2	4	5	6	7
4/3	**4/3**	**6/5**	35/24	4/3	81/56	(4/3)

NEW SCALE

1/1 28/27 16/15 4/3 3/2 14/9 16/9 2/1
28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 8/7 · 9/8

Archytas's diatonic and yields a mixed scale, half enharmonic and half diatonic. Further application of this principle produces additional scales until the original sequence is restored. Each of these scales could be modally (cyclically) permuted as well.

Wilson derives a number of the thats of North Indian ragas by operating on various arrangements of the tetrachords 256/243 · 9/8 · 9/8, 16/15 · 9/8 · 10/9, 28/27 · 8/7 · 9/8, 16/15 · 135/128 · 32/27, and 10/9 · 10/9 · 27/25. He then generates analogs of these scales from other tetrachords, including those with undecimal intervals.

In his 1987 paper, Wilson described a complementary technique of modulation ("The Purvi Modulations"). This technique makes use of the fact that at least one of the fourths differs greatly in size from the rest. The exceptional fourth may be abstracted from the linear fourth sequence and interpolated between successive pairs to generate derived scales. At the end of seven such interpolations, the linear sequence is cyclically permuted by one position and the process of interpolation continued. After 42 steps the

6-16. Modulations by sequential fourths. This example begins with the Dorian mode for consistency with other sections of this treatise. The sizes of the fourths range from 6/5 (316 cents) to 35/24 (653 cents). In the original sequence the exceptional fourth is in boldface. In the rotated sequence the scale has been modally permuted to separate the exceptional fourth (in boldface) from the rest. In the first modulated sequence the 6/5 (in boldface) has been interpolated between fourths 7 and 1 of the original series. In the second modulated sequence the 6/5 (in boldface) has been interpolated between fourths 3 and 4 of the original series. The new tetrachord is Archytas's diatonic.

ORIGINAL SCALE

1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1

28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35 · 5/4

	FOURTHS	SIZE
1.	1/1 TO 4/3	4/3
2.	4/3 TO 8/5	6/5
3.	8/5 TO 16/15	4/3
4.	16/15 TO 14/9	35/24
5.	14/9 TO 28/27	4/3
6.	28/27 TO 3/2	81/56
7.	3/2 TO 2/1	4/3

ORIGINAL SEQUENCE

1	2	3	4	5	6	7
4/3	**6/5**	4/3	35/24	4/3	81/56	4/3

ROTATED SEQUENCE

3	4	5	6	7	1	2
4/3	35/24	4/3	81/56	4/3	4/3	**6/5**

NEW SCALE

1/1 5/4 35/27 4/3 5/3 15/8 35/18 2/1

5/4 · 28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35

THE LINEAR SEQUENCE OF FOURTHS

4/3 35/24 4/3 81/56 4/3 4/3

MODULATED SEQUENCE 1

2	3	4	5	6	7	1
6/5	4/3	35/24	4/3	81/56	4/3	4/3

NEW SCALE 1

1/1 9/8 7/6 6/5 3/2 14/9 8/5 2/1

9/8 · 28/27 · 36/35 · 5/4 · 28/27 · 36/35 · 5/4

MODULATED SEQUENCE 2

3	2	4	5	6	7	1
4/3	**6/5**	35/24	4/3	81/56	4/3	4/3

NEW SCALE 2

1/1 9/8 7/6 4/3 3/2 14/9 8/5 2/1

9/8 · 28/27 · 8/7 · 9/8 · 28/27 · 36/35 · 5/4

original scale is restored, but transposed to a new and remote key. Wilson also provides an alternate derivation which better brings out the transpositional character of the process. In this case the linear sequence of non-exceptional fourths is tandemly duplicated to form a series of indefinite extent. Successive overlapping 6-unit segments of this series are appended with the exceptional fourth to form octave scales. After seven operations, the sequence repeats with a new mode of the original scale. The process is illustrated in 6-16.

Non-traditional scale forms

In the remainder of this chapter, some non-traditional approaches to scale construction from tetrachordal modules will be presented. These approaches are presented as alternatives to the historical modes and other types of scales which were discussed in the earlier parts of this chapter.

The first group of non-standard tetrachordal scales is generated by combining a given tetrachord with an identical one transposed by one of its own structural intervals or the inversion of one of these intervals (6-17). This process yields 7-tone scales, including three of the traditional modes if the interval is 4/3, 3/2, or with a slight stretching of the concept, 9/8 and 3/2 together. The other tetrachordal complexes, however, are quite different from the historical modes.

6-17. *Complexes of one tetrachordal form.*

1. TRANSPOSITION BY *a*
1/1 *a b* 2*a ab* 4/3 4*a*/3 2/1

2. TRANSPOSITION BY *b*
1/1 *a b ab* 2*b* 4/3 4*b*/3 2/1

3. TRANSPOSITION BY 4/3, MIXOLYDIAN
1/1 *a b* 4/3 4*a*/3 4*b*/3 16/9 2/1

4. TRANSPOSITION BY 3/2, DORIAN
1/1 *a b* 4/3 3/2 3*a*/2 3*b*/2 2/1

5. TRANSPOSITION BY 2/*b*
1/1 *a b* 4/3 2/*b a*/*b* 4/3*b* 2/1

6. TRANSPOSITION BY 2/*a*
1/1 *a b* 4/3 2/*a b*/*a* 4/3*a* 2/1

7. TRANSPOSITION BY 9/8 & 3/2, HYPODORIAN
1/1 9/8 9*a*/8 9*b*/8 3/2 3*a*/2 3*b*/2 2/1

8. TRANSPOSITION BY 4/3*b*
1/1 *a b* 4/3*b* 4/3 4*a*/3*b* 16/9*b* 2/1

9. TRANSPOSITION BY 4/3*a*
1/1 *a b* 4/3*a* 4/3 4*b*/3*a* 16/9*a* 2/1

10. TRANSPOSITION BY *a*/*b*
1/1 *a*2/*b a b* 4*a*/3*b* 4/3 *a*/*b* 2/1

11. TRANSPOSITION BY *b*/*a*
1/1 *b*/*a a b b*2/*a* 4/3 4*b*/3*a* 2/1

6-18 provides examples of the resulting scales when the generating tetrachord is Archytas's enharmonic, 28/27 · 36/35 · 5/4. In this case interval *a* equals 28/27 and *b* is 16/15 (28/27 · 36/35).

As some of these tetrachordal complexes have large gaps, one might try combining two of them, one built upwards from 1/1 and the other downwards from 2/1 to create a more even scale, though there are precedents for such gapped scales, i.e., the Mixolydian harmonia (6-4). While the normal ascending or prime form of the tetrachord—the one whose intervals are in the order of smallest, medium and largest—is used to demonstrate the technique, any of the six permutations would serve equally well. In fact, Archytas's enharmonic and diatonic genera are not strictly of this form as 28/27 is larger than 36/35 and 8/7 is wider than 9/8.

The next class of tetrachordal complexes are those composed of a tetrachord and its inverted form. 6-19 lists some simple examples of this approach; 6-20 lists the resulting notes in Archytas's enharmonic tuning. These scales have six, seven, or eight tones.

6-18. *Complexes of the prime form of Archytas's enharmonic.*

1. TRANSPOSITION BY *a*
1/1 28/27 16/15 784/729 448/405 4/3 112/81 2/1
0 63 112 126 175 498 561 1200

2. TRANSPOSITION BY *b*
1/1 28/27 16/15 448/405 256/225 4/3 64/32 2/1
0 63 112 175 223 498 610 1200

3. TRANSPOSITION BY 4/3 MIXOLYDIAN
1/1 28/27 16/15 4/3 112/81 64/45 16/9 2/1
0 63 112 498 561 610 996 1200

4. TRANSPOSITION BY 3/2, DORIAN
1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1
0 63 112 498 702 765 814 1200

5. TRANSPOSITION BY 2/*b*
1/1 28/27 16/15 5/4 4/3 15/8 35/18 2/1
0 63 112 386 498 1088 1151 1200

6. TRANSPOSITION BY 2/*a*
1/1 36/35 28/27 16/15 9/7 4/3 27/14 2/1
0 49 63 112 435 498 1137 1200

7. TRANSPOSITION BY 9/8 & 3/2, HYPODORIAN
1/1 9/8 7/6 6/5 3/2 14/9 8/5 2/1
0 204 267 316 702 765 814 1200

8. TRANSPOSITION BY 4/3*b*
1/1 28/27 16/15 5/4 35/27 4/3 5/3 2/1
0 63 112 386 449 498 884 1200

9. TRANSPOSITION BY 4/3*a*
1/1 28/27 16/15 9/7 4/3 48/35 12/7 2/1
0 63 112 435 498 547 933 1200

10. TRANSPOSITION BY *a*/*b*
1/1 245/243 28/27 16/15 35/27 4/3 35/18 2/1
0 14 63 112 449 498 1151 1200

11. TRANSPOSITION BY *b*/*a*
1/1 36/35 28/27 16/15 192/175 4/3 48/35 2/1
0 49 63 112 161 498 561 1200

6-19. *Simple complexes of prime and inverted forms. Two versions of the pseudo- (Ψ-) Hypodorian mode are shown to illustrate the effect of reversing the placement of the prime and inverted forms. The two scales are not modes of each other.*

1. TRANSPOSITION AND INVERSION BY a, 6 TONES, A HEXANY

1/1 a b $4a/3b$ 4/3 $4a/3$ 2/1

2. TRANSPOSITION AND INVERSION BY b, 6 TONES, A HEXANY

1/1 a b 4/3 $4b/3a$ $4b/3$ 2/1

3. TRANSPOSITION AND INVERSION BY 4/3, 7 TONES, ψ-MIXOLYDIAN

1/1 a b 4/3 $16/9b$ $16/9a$ 16/9 2/1

4. TRANSPOSITION AND INVERSION BY 3/2, 7 TONES, ψ-DORIAN

1/1 a b 4/3 3/2 $2/b$ $2/a$ 2/1

5. TRANSPOSITION AND INVERSION BY $2/b$, 8 TONES, AN OCTONY

1/1 a b 4/3 $2/b$ $4/3b^2$ $4/3ab$ $4/3b$ 2/1

6. TRANSPOSITION AND INVERSION BY $2/a$, 8 TONES, AN OCTONY

1/1 a b 4/3 $2/a$ $4/3a^2$ $4/3ab$ $4/3a$ 2/1

7. TRANSPOSITION AND INVERSION BY 9/8 & 3/2, 7 TONES, ψ-HYPODORIAN 1

1/1 9/8 $3/2b$ $3/2a$ 3/2 $3a/2$ $3b/2$ 2/1

8. TRANSPOSITION AND INVERSION BY 9/8 & 3/2, 7 TONES, ψ-HYPODORIAN 2

1/1 9/8 $9a/8$ $9b/8$ 3/2 $2/b$ $2/a$ 2/1

9. TRANSPOSITION AND INVERSION BY 1/1, 6 TONES, A HEXANY

1/1 a b $4/3b$ $4/3a$ 4/3 2/1

10. TRANSPOSITION AND INVERSION BY $4/3b$, 8 TONES, AN OCTONY

1/1 a b $4/3b$ 4/3 $16/9b^2$ $16/9ab$ $16/9b$ 2/1

11. TRANSPOSITION AND INVERSION BY $4/3a$, 8 TONES, AN OCTONY

1/1 a b $4/3a$ 4/3 $16/9ab$ $16/9a^2$ $16/9a$ 2/1

12. TETRACHORDAL HEXANY, 6 TONES, A-MODE

1/1 b/a b $4/3a$ 4/3 $4b/3a$ 2/1

13. EULER'S GENUS MUSICUM, 8 TONES, AN OCTONY

1/1 a b ab 4/3 $4a/3$ $4b/3$ $4ab/3$ 2/1

14. TRANSPOSITION AND INVERSION BY B/A, 8 TONES, AN OCTONY

1/1 b/a a b $4/3a$ 4/3 $4b/3a2$ $4b/3a$ 2/1

15. TRANSPOSITION AND INVERSION BY A/B, 8 TONES, AN OCTONY

1/1 a b $4a/3b^2$ $4/3b$ $4a/3b$ 4/3 a/b 2/1

6-20. Simple complexes of the prime and inverted forms of Archytas's enharmonic, in ratios and cents. Two versions of the Ψ-hypodorian mode are shown to illustrate the effect of reversing the placement of the prime and inverted forms. The two scales are not modes of each other.

The 7-tone scales are analogous to the traditional Greek modes, whose names are appropriated with a prefixed Ψ (for pseudo) to indicate their relationship to the prototypes. Although these 7-tone scales were produced by pairing a tetrachord with its inversion, in principle any two dissimilar permutations would yield a heptatonic scale. This degree of flexibility is not true of the 6- and 8-tone types for which the pairing of prime and inverted forms is mandatory.

1. TRANSPOSITION AND INVERSION BY *a*, 6 TONES, A HEXANY
1/1 28/27 16/15 35/27 4/3 112/81 2/1
0 63 112 449 498 561 1200

2. TRANSPOSITION AND INVERSION BY *b*, 6 TONES, A HEXANY
1/1 28/27 16/15 4/3 48/35 64/45 2/1
0 63 112 498 547 610 1200

3. TRANSPOSITION AND INVERSION BY 4/3, 7 TONES, Ψ-MIXOLYDIAN
1/1 28/27 16/15 4/3 5/3 12/7 16/9 2/1
0 63 112 498 884 933 996 1200

4. TRANSPOSITION AND INVERSION BY 3/2, 7 TONES, Ψ-DORIAN
1/1 28/27 16/15 4/3 3/2 15/8 27/14 2/1
0 63 112 498 702 1088 1137 1200

5. TRANSPOSITION AND INVERSION BY 2/*b*, 8 TONES, AN OCTONY
1/1 28/27 16/15 75/64 135/112 5/4 4/3 15/8 2/1
0 63 112 275 323 386 498 1088 1200

6. TRANSPOSITION AND INVERSION BY 2/*a*, 8 TONES, AN OCTONY
1/1 28/27 16/15 135/112 243/196 9/7 4/3 27/14 2/1
0 63 112 323 372 435 498 1137 1200

7. TRANSPOSITION AND INVERSION BY 9/8 & 3/2, 7 TONES, Ψ-HYPODORIAN 1
1/1 9/8 45/32 81/56 3/2 14/9 8/5 2/1
0 204 590 639 702 765 814 1200

8. TRANSPOSITION AND INVERSION BY 9/8 & 3/2, 7 TONES, Ψ-HYPODORIAN 2
1/1 9/8 7/6 6/5 3/2 15/8 27/14 2/1
0 204 267 316 702 1088 1137 1200

9. TRANSPOSITION AND INVERSION BY 1/1, 6 TONES, A HEXANY
1/1 28/27 16/15 5/4 9/7 4/3 2/1
0 63 112 386 435 498 1200

10. TRANSPOSITION AND INVERSION BY 4/3*b*, 8 TONES, AN OCTONY
1/1 28/27 16/15 5/4 4/3 25/16 45/28 5/3 2/1
0 63 112 386 498 773 821 884 1200

11. TRANSPOSITION AND INVERSION BY 4/3*a*, 8 TONES, AN OCTONY
1/1 28/27 16/15 9/7 4/3 45/28 81/49 12/7 2/1
0 63 112 435 498 821 870 933 1200

12. TETRACHORDAL HEXANY, 6 TONES, A-MODE
1/1 36/35 16/15 9/7 4/3 48/35 2/1
0 49 112 435 498 547 1200

13. EULER'S GENUS MUSICUM, 8 TONES, AN OCTONY
1/1 28/27 16/15 448/405 4/3 112/81 64/45 1792/1215 2/1
0 63 112 175 498 561 610 673 1200

14. TRANSPOSITION AND INVERSION BY *b/a*, 8 TONES, AN OCTONY
1/1 36/35 28/27 16/15 9/7 324/245 4/3 48/35 2/1
0 49 63 112 435 484 498 561 1200

15. TRANSPOSITION AND INVERSION BY *a/b*, 8 TONES, AN OCTONY
1/1 28/27 16/15 175/144 5/4 35/27 4/3 35/18 2/1
0 63 112 338 386 449 498 1151 1200

Tetrachordal hexanies

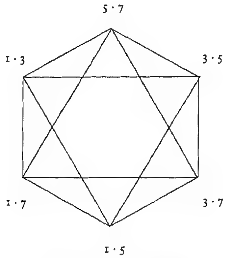
The 6-tone complexes are of greater theoretical interest than either the seven or 8-tone scales. Because of their quasi-symmetrical melodic structure, which is a circular permutation of the interval sequence *c b a b c d* (*a*, *b*, *c*, and *d* not necessarily different intervals), they are members of a class of scales discovered by Ervin Wilson and termed *combination product sets* (Wilson 1989; Chalmers and Wilson 1982; Wilson, personal communication). The same structure results if interval *a* is replaced with interval *d* and intervals *b* and *c* are exchanged. A combination product set of six tones is called a *hexany* by Wilson.

The notes of the hexany are the melodic expansion of the intervals of a generating tetrad or tetrachord. They are obtained by forming the six binary products of the four elements of the generator. If these four elements are labelled x, y, z, and w, the resulting notes are $x \cdot y$, $x \cdot z$, $x \cdot w$, $y \cdot z$, $y \cdot w$, and $w \cdot z$. In the case where the generator is the dominant seventh tetrad, 1/1 5/4 3/2 7/4, written in factor form as 1 3 5 7, the resulting hexany is that of 6-21, where it has been mapped over the vertices of a regular octahedron. This diagram has been named a "hexagram" by Wilson.

It is convenient to choose one of these tones and transpose the scale so that it starts on this note. The note 3 · 5 has been selected in 6-21. This note, however, should not be considered as the tonic of the scale; the combination product sets are harmonically symmetrical, polytonal sets with virtual or implicit tonics which are not tones of the scale. Although the hexany is partitionable into a set of rooted triads (see below), the global 1/1 for the whole set is not a note of the scale. In this sense, combination product sets are a type of atonal or non-centric musical structure in just intonation.

The four elements of the generator are related to the melodic intervals as $x = 1/1$, $y = b$, $z = b \cdot c$, and $w = a \cdot b^2 \cdot c$, although the actual tones may have to be transposed or circularly permuted to make this relationship clearer.

6-21. *The 1 3 5 7 tetradic hexany. The factor 1 may be omitted from the three tones which contain it. This diagram was invented by Ervin Wilson and represents the six tones of the hexany mapped over the six vertices of the regular octahedron (Wilson 1989). Each triangular face is an essential consonant chord of the hexany harmonic system and every pair of tones separated by a principal diagonal is a dissonance. The keynote is 3·5.*



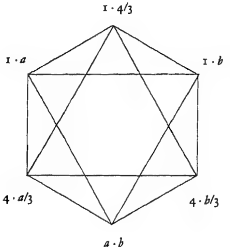
NOTES AND INTERVALS OF HEXANY

1/1		7/6		4/3		7/5		8/5		28/15		2/1
	7/6		8/7		21/20		8/7		7/6		15/14	
	c		*b*		*a*		*b*		*c*		*d*	

6-22. *Consonant chords of the 1 3 5 7 hexany.*

CHORD	HARMONIC	SUBHARMONIC
1 3 5	1·7 3·7 5·7	3·5 1·5 1·3
1 3 7	1·5 3·5 5·7	3·7 1·7 1·3
1 5 7	1·3 3·5 3·7	5·7 1·7 1·5
3 5 7	1·3 1·5 1·7	5·7 3·7 3·5

6-23. *The tetrachordal hexany. Based on the generating tetrad 1/1* a b *4/3. After transposition by* a, *it is equivalent to complex 12 of 6-19 and 6-20.*



NOTES AND INTERVALS OF HEXANY

1/1	*b/a*	*b*	4/3*a*	4/3	4*b*/3*a*	2/1
1/1	36/35	16/15	9/7	4/3	48/35	2/1
36/35	28/27	135/112	28/27	36/35	35/24	
c	*b*	*a*	*b*	*c*	*d*	

The six tones of the hexany may be partitioned into four sets of three tones and their inversions. In the hexagram or octahedral representation, the 3-tone sets appear as triangular faces or facets. The triads of 6-21 are tabulated in 6-22. These chords are the essential consonant chords of the hexany, and all chords containing pairs of tones separated by diagonals are considered dissonant.

Armed with this background, one can now proceed to the generation of hexanies from tetrachords. Starting with the tetrachord 1/1 *a b* 4/3 (the generator of complex 12 in 6-19), the generative process and the relationships between the notes may be seen in 6-23. Archytas's enharmonic (1/1 28/27 16/15 4/3; 28/27 · 36/35 · 5/4; *a* = 28/27, *b* = 16/15) is the specific generator (see also 6-20, complex 12). This hexany has been transposed so that the starting note 1·*a* is 1/1.

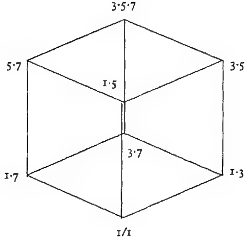
Tetrachordal hexanies are melodic developments of the basic intervals rather than harmonic expansions of tetrads. The triangular faces of tetrachordal hexanies are 2-interval subsets of the three intervals of the original tetrachord. Since this is basically a melodic development, the faces will be referred to as essential subsets rather than consonant chords. (For the same reason, the terms *harmonic* and *subharmonic* are replaced by *prime* and *inverted.*) These hexanies may be partitioned into essential subsets as shown in 6-24.

The generator of complex 1 of 6-19 and 6-20 (inversion and transposition by *a*) is the permuted tetrachord 1/1 *b/a b* 4/3 (1/1 36/35 16/15 4/3; 36/35 · 28/27 · 5/4; *a* = 36/35, *b* = 16/15). The generators of complexes 2 and 9 are 1/1 *b/a b* 4*b*/3*a* (1/1 36/35 16/15 48/35; 36/35 · 28/27 · 9/7) and

6-24. *Essential subsets of the hexanies based on the tetrachords 1/1* a b *4/3 and 1/1 28/27 16/15 4/3 (Archytas's enharmonic). For the sake of clarity, the factor 1 (1/1) has been omitted from 1*·a, *1*·b, *and 1·4/3. The · signs are also deleted. Both hexanies are given in their untransposed forms.*

SUBSET	PRIME	INVERTED
1/1 *a b*	4/3 4*a*/3 4*b*/3	*ab b a*
1/1 *a* 4/3	*b ab* 4*b*/3	4*a*/3 4/3 *a*
1/1 *b* 4/3	*a ab* 4*a*/3	4B/3 4/3 *b*
a b 4/3	*a b* 4/3	4*b*/3 4*a*/3 *ab*
1/1 28/27 16/15	4/3 112/81 64/45	448/405 16/15 28/27
1/1 28/27 4/3	16/15 448/405 64/45	112/81 4/3 28/27
1/1 16/15 4/3	28/27 448/405 112/81	64/45 4/3 16/15
28/27 16/15 4/3	28/27 16/15 4/3	64/45 112/81 448/405

6-25. *The 1 3 5 7 tetradic octony. This structure is also an Euler's genus (Fokker 1966; Euler 1739).*



6-26. *Essential chords of the 1 3 5 7 tetradic octony.*

CHORD	PRIME	INVERTED
FACE	1/1 1·3 1·5 3·7	5·7 1·5 3·5 3·5·7
	1/1 1·5 1·5 3·5	1·7 5·7 3·7 3·5·7
	1/1 1·7 1·5 5·7	1·3 3·7 3·5 3·5·7
VERTEX	1/1 1·3 1·5 1·7	3·5·7 3·5 3·7 5·7
	1·7 5·7 1/1 3·7	1·5 1·3 3·5 3·5·7
	1·5 1/1 5·7 3·5	3·7 1·3 1·7 3·5·7
	1·3 3·5 3·7 1/1	5·7 1·7 1·5 3·5·7
DIAGONAL	1/1 5·7 3·5 3·7	3·5·7 1·5 1·3 1·7

1/1 *b/a b* 4/3*a* (1/1 36/35 16/15 9/7; 36/35 · 28/27 · 135/112) respectively. In these hexanies, the tetrachordal generators are bounded by augmented and diminished fourths rather than 4/3's, but the subset relations are analogous to those with perfect fourths.

Tetrachordal Euler genera

The 8-tone complexes represent a different type of scale which may be called an *interval symmetric set* (Chalmers and Wilson 1982; Chalmers 1983). These scales have the melodic sequence *d c b a b c d e* which is homologous to the *c b a b c d* sequence of the hexany. However, these 8-tone scales lack some of the harmonic and structural symmetries that characterize the combination product sets.

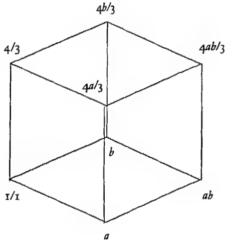
Wilson has pointed out that these sets are members of a large class of scales invented by Leonhard Euler in the eighteenth century and publicized by A. D. Fokker (Wilson, personal communication). While they have been given the generic name of octony in analogy with the hexany and other combination product sets, the terms Euler genus or Euler-Fokker genus would seem to have priority as collective names (Fokker 1966; Rasch 1987).

The generation of an octony from the 1 3 5 7 tetrad is shown in 6-25. In this representation, the eight tones have been mapped over the vertices of a cube. This diagram may be called an "octagram." The octony may also be partitioned into inversionally paired subsets, but the chords are generally more complex than those of hexanies derived from the same generator (6-26). Chords considered as the essential consonances of a harmonic system based on the octony appear not only as faces (face chords), but also as vertices with their three nearest neighbors connected by edges (vertex chords) or by face diagonals (vertex-diagonal chords) (Chalmers 1983). Essential dissonances are any chords containing a pair of tones separated by a principal diagonal of the cube.

With the exception of the generator itself and its inversion, each of the 4-note chords consists of the union of a harmonic and subharmonic triad of the form 1/1 *x y* and *x y x·y*. An analogous chord in traditional theory is the major triad with the major seventh added, 1/1 5/4 3/2 15/8, which could be construed as a major triad on 1/1 fused with a minor triad on 5/4.

As in the case of the hexany, octonies may be constructed from tetrachords and their inversions (6-27). The clearest example is complex 13 of

6-27. *The tetrachordal octony. This 8-tone Euler's genus is generated from the generalized tetrachord* 1/1 a b *4/3.*



6-28. *Essential subsets of the tetrachordal octonies 1/1* a b *4/3 and 1/1 28/27 16/15 4/3 (Archytas's enharmonic). The term* essential subset *rather than* consonant chord *is employed as the tetrachordal octony is primarily a melodic structure.*

6-18 which is generated by the tetrachord 1/1 *a b* 4/3. Its subset structure is shown in 6-28. The generating tetrachord and its inversion appear as face chords. The other chords are more complex intervallic sets. Like the hexany above, the octony should be viewed as a melodic rather than a harmonic development of the tetrachord.

The other 8-tone complexes of 6-19 are also octonies. The complexes generated from Archytas's enharmonic genus are listed in 6-20.

Tetrachordal diamonds

The next group of non-traditional tetrachordal scales is even more complex than the previous constructions. The first of these are based on the Partch diamond (Partch [1949] 1974) which is an interlocking matrix of harmonic

NOTE AND INTERVALS OF OCTONY

1/1	*a*	*b*	*ab*	4/3	*4a/3*	*4b/3*	*4ab/3*	2/1
1/1	28/27	16/15	448/405	4/3	112/81	64/45	1792/1215	2/1

28/27	36/35	28/27	135/112	28/27	36/35	28/27	1215/896
d	*c*	*b*	*a*	*b*	*c*	*d*	*e*

SUBSET	PRIME	INVERTED
FACE	1/1 4/3 *4a/3a*	*4ab/3 ab b 4b/3*
	1/1 4/3 *4b/3b*	*4ab/3 ab a 4a/3*
	1/1 *a b ab*	*4ab/3 4b/3 4a/3* 4/3
VERTEX	1/1 *a b* 4/3	*4ab/3 4b/3 4a/3 ab*
	4/3 1/1 *4a/3*	*4b/3 ab 4ab/3 b a*
	4a/3 a 4/3	*4ab/3b 4b/3 ab* 1/1
	4b/3 4/3 *b*	*4ab/3a ab 4a/3* 1/1
DIAGONAL	1/1 *4b/3 4a/3 ab*	*4ab/3 a b* 4/3
FACE	1/1 4/3 112/81 28/27	1792/1215 448/405 16/15 64/45
	1/1 4/3 64/45 16/15	1792/1215 448/405 28/27 112/81
	1/1 28/27 16/15 448/405	1792/1215 64/45 112/81 4/3
VERTEX	1/1 28/27 16/15 4/3	1792/1215 64/45 112/81 448/405
	4/3 1/1 112/81 64/45	448/405 1792/1215 16/15 28/27
	112/81 28/27 4/3 16/15	64/45 448/405 1/1 1792/1215
	64/45 4/3 16/15 28/27	448/405 112/81 1/1 1792/1215
DIAGONAL	1/1 64/45 112/81 448/405	1792/1215 28/27 16/15 4/3

chords built on roots that are the elements of the corresponding subharmonic ones. An example of what is called a *5-limit* diamond may be seen in 6-30. This example has been constructed from harmonic 1 3 5; major triads and subharmonic 1 3 5; or minor triads. The structure is referred to as having a 5-limit because the largest prime number appearing among its ratios is five. Diamonds, however, may be constructed from any chord or scale of any cardinality, magnitude, or limit.

The simplest of the tetrachordal diamonds consists of ascending tetrachords erected on the notes of their inversions. Either the octave or the 4/3 (numbers 1 and 2 of 6-29) may be used as the interval of identity in the diamond. In the latter case, the resulting structure is one of the rare examples of musical scales in which the octave is not the interval of equivalence.

The second group of diamond-like complexes employs entire heptatonic scales in place of triads or tetrachords as structural elements. Four examples are given, all derived from scales of the Dorian or Ψ-Dorian type in which prime or inverted tetrachords appear in either or both positions relative to the central disjunctive tone (6-29, numbers 2, 4, 5; and 6-34). The prime-prime and inverted-inverted diamonds have prime or inverted tetrachords in both halves of the generating scales. Because of the inversional symmetry

6-29. *Tetrachordal diamonds. The octave modular tetrachordal diamond in Archytas's enharmonic tuning is shown in 6-33.*

1. THIRTEEN TONE OCTAVE MODULAR DIAMOND

1/1 *b/a a b* 4/3*b* 4/3*a* 4/3 3/2 3*a*/2 3*b*/2 2/*b* 2/*a a/b* 2/1

2. EIGHT TONE FOURTH MODULAR DIAMOND

1/1 *a b* 4/3*b* 4/3*a* 4*a*/3*b* 4/3 4*b*/3*a*

3. PRIME–PRIME AND INVERTED–INVERTED HEPTATONIC DIAMONDS, 27 TONES

1/1 *b/a a* B 9/8 9*a*/8 9*b*/8 4/3*b* 4/3*a* 4*a*/3*b* 4/3 4*b*/3*a* 4*a*/3 3/2*b* 4*b*/3 3/2*a* 3*a*/2*b* 3/2 3*b*/2*a* 3*a*/2 3*b*/2 16/9*b* 16/9*a* 16/9 2/*b* 2/*a a/b* 2/1

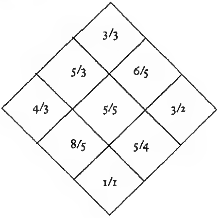
4. PRIME-INVERTED HEPTATONIC DIAMOND, 25 TONES

1/1 *b/a a b* a^2 *ab* 9/8 b^2 4/3*b* 4/3*a* 4/3 4*a*/3 3/2*b* 4*b*/33/2*a* 3/2 3*a*/2 3*b*/2 $2/b^2$ 16/9 2/*ab* $2/a^2$ 2/*b* 2/*a a/b* 2/1

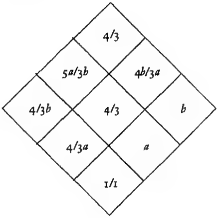
5. INVERTED-PRIME HEPTATONIC DIAMOND, 25 TONES

1/1 *b/a a b* 9/8 9*a*/8 9*b*/8 $9a^2/8$ 9*ab*/8 4/3*b* $9b^2/8$ 4/3*a*4/3 3/2 3*a*/2 $16/9b^2$ 3*b*/2 16/9*ab* $16/9a^2$ 16/9*b* 16/9*a* 16/92/*b* 2/*a a/b* 2/1

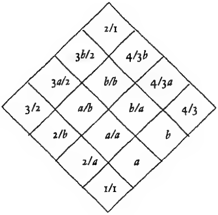
6-30. *Five-limit Partch diamond, after "The Incipient Tonality Diamond" (Partch [1949] 1974, 110). Based on the 1 3 5 major triad 1/1 5/4 3/2 and its inversion, the subharmonic 1 3 5 minor triad 2/1 8/5 4/3.*



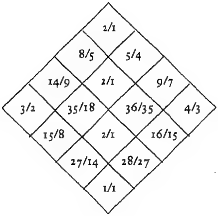
6-31. *Eight tone fourth modular diamond. Based on the tetrachord 1/1 a b 4/3, with 4/3 as the interval of equivalence.*



6-32. *Thirteen-tone octave modular tetrachordal diamond.*



6-33. *Thirteen-tone octave modular tetrachordal diamond based on Archytas's enharmonic genus.*



of the diamond, both scales are identical. The prime-inverted and inverted-prime diamonds are constructed from the corresponding tetrachordal forms and are non-equivalent scales, as in general, tetrachords are not inversionally symmetrical intervallic sequences. 6-35 and 6-36 show examples of these diamonds based on Archytas's enharmonic genus and its inversion.

6-34. *Tetrachordal heptatonic diamonds. These tables may be rotated 45 degrees clockwise to bring the diagonal of 2/1's into vertical position and compared to figures 6-30–33. The scale derived from the prime form of the tetrachord is seen in the rightmost column and its inversion in the bottom row.*

Stellated tetrachordal hexanies

The last of the non-traditional tetrachordal complexes to be discussed are two examples of stellated hexanies. Hexanies may be stellated by adding the eight tones which complete the partial tetrad or tetrachord on each face (Wilson 1989; Chalmers and Wilson 1982). The result is a complex of four

PRIME–PRIME

2/1	b/a	b	9b/8	3/2	3b/2a	3b/2
a/b	2/1	a	9a/8	3a/2b	3/2	3a/2
2/b	2/a	2/1	9/8	3/2b	3/2a	3/2
16/9b	16/9a	16/9	2/1	4/3b	4/3a	4/3
4/3	4b/3a	4b/3	3b/2	2/1	b/a	b
4a/3b	4/3	4a/3	3a/2	a/b	2/1	a
4/3b	4/3a	4/3	3/2	2/b	2/a	1/1

PRIME–INVERTED

2/1	b/a	4/3a	3/2a	2/ab	2/a2	2/a
a/b	2/1	4/3b	3/2b	2/b2	2/ab	2/b
3a/2	3b/2	2/1	9/8	3/2b	3/2a	3/2
4a/3	4b/3	16/9	2/1	4/3b	4/3a	4/3
ab	b2	4b/3	3b/2	2/1	b/a	b
a2	ab	4a/3	3a/2	a/b	2/1	a
a	b	4/3	3/2	2/b	2/a	1/1

INVERTED–INVERTED

2/1	b/a	4/3b	3/2a	3/2	3b/2a	2/a
a/b	2/1	4/3b	3/2b	3a/2b	3/2	2/b
3a/2	3b/2	2/1	9/8	9a/b	9b/a	3/2
4a/3	4b/3	16/9	2/1	a	b	4/3
4/3	4b/3a	16/9a	2/a	2/1	b/a	4/3a
4a/3b	4/3	16/9b	2/b	a/b	2/1	4/3b
a	b	4/3	3/2	3a/2	3b/2	1/1

INVERTED–PRIME

2/1	b/a	b	9b/8	9ab/8	9b2/8	3b/2
a/b	2/1	a	9a/8	9a2/8	9ab/8	3a/2
2/b	2/a	2/1	9/8	9a/8	9b/8	3/2
16/9b	16/9a	16/9	2/1	a	b	4/3
16/9ab	16/9a2	16/9a	2/a	2/1	b/a	4/3a
16/9b2	16/9ab	16/9b	2/b	a/b	2/1	4/3b
4/3b	4/3a	4/3	3/2	3a/2	3b/2	1/1

prime and four inverted tetrachords with a total of fourteen tones, though certain genera may produce degenerate complexes with fewer than 14 different notes. Wilson has variously termed these structures "mandalas" from their appearance in certain projections, and "tetradekanies" or "dekatesseranies" from their fourteen tones. Their topology is that of Kepler's stella octangula, an 8-pointed star-polyhedron (Coxeter 1973; Cundy and Rollett 1961).

The prime form of the tetrachord 1/1 *a b* 4/3 generates the hexany tones *a*, *b*, 4/3, 4*a*/3, 4*b*/3 and *ab* (*a* = 1/1·*a* or 1·*a*, etc.). This hexany is equivalent

6-35. *Tetrachordal diamonds based on Archytas's enharmonic, in ratios and cents.*

13-TONE OCTAVE MODULAR DIAMOND

1/1	36/35	28/27	16/15	5/4	9/7	4/3	3/2	14/9	8/5	15/8	27/14	35/18	2/1
0	49	63	112	386	435	498	702	765	814	1088	1137	1151	1200

8-TONE TETRACHORD MODULAR DIAMOND

1/1	28/27	16/15	5/4	9/7	35/27	4/3	48/35
0	63	112	386	435	449	498	547

PRIME-PRIME AND INVERTED-INVERTED HEPTATONIC DIAMONDS, 27 TONES

1/1	36/35	28/27	16/15	9/8	7/6	6/5	5/4	9/7	35/27	4/3	48/35	112/81
0	49	63	112	204	267	316	386	435	449	498	547	561

45/32	64/45	81/56	35/24	3/2	54/35	14/9	8/5	5/3	12/7	16/9	15/8	27/14	35/18	2/1
590	610	639	653	702	751	765	814	884	933	996	1088	1137	1151	1200

PRIME-INVERTED HEPTATONIC DIAMOND, 25 TONES

1/1	36/35	28/27	16/15	784/729	448/405	9/8	256/225	5/4	9/7	4/3	112/81
0	49	63	112	126	175	204	223	386	435	498	561

45/32	64/45	81/56	3/2	14/9	8/5	225/128	16/9	405/224	729/392	15/8	27/14	35/18	2/1
590	610	639	702	765	814	977	996	1025	1074	1088	1137	1151	1200

INVERTED-PRIME HEPTATONIC DIAMOND, 25 TONES

1/1	36/35	28/27	16/15	9/8	7/6	6/5	98/81	56/45	5/4	32/25
0	49	63	112	204	267	316	330	379	386	427

9/7	4/3	3/2	14/9	25/16	8/5	45/28	81/49	5/3	12/7	15/8	27/14	35/18	2/1	16/9
435	498	702	765	773	814	821	870	884	933	996	1088	1137	1151	1200

6-36. *Tetrachordal heptatonic diamonds based on Archytas's enharmonic. The generating tetrachords are 1/1 5/4 9/7 4/3 and 1/1 28/27 16/15 4/3.*

PRIME–PRIME

2/1	36/35	16/15	6/5	3/2	54/35	8/5
35/18	2/1	28/27	7/6	35/24	3/2	14/9
15/8	27/14	2/1	9/8	45/32	81/56	3/2
5/3	12/7	16/9	2/1	5/4	9/7	4/3
4/3	48/35	64/45	8/5	2/1	36/35	16/15
35/27	4/3	112/81	14/9	35/18	2/1	28/27
5/4	9/7	4/3	3/2	15/8	27/14	1/1

PRIME–INVERTED

2/1	36/35	9/7	81/56	405/224	729/392	27/14
35/18	2/1	5/4	45/32	225/128	405/224	15/8
14/9	8/5	2/1	9/8	45/32	81/56	3/2
112/81	64/45	16/9	2/1	5/4	9/7	4/3
448/405	256/225	64/45	8/5	2/1	36/35	16/15
784/729	448/405	112/81	14/9	35/18	2/1	28/27
28/27	16/15	4/3	3/2	15/8	27/14	1/1

INVERTED–INVERTED

2/1	36/35	9/7	81/56	3/2	54/35	27/14
35/18	2/1	5/4	45/32	35/24	3/2	15/8
14/9	8/5	2/1	9/8	7/6	6/5	3/2
112/81	64/45	16/9	2/1	28/27	16/15	4/3
4/3	48/35	12/7	27/14	2/1	36/35	9/7
35/27	4/3	5/3	15/8	35/18	2/1	5/4
28/27	16/15	4/3	3/2	14/9	8/5	1/1

INVERTED–PRIME

2/1	36/35	16/15	6/5	56/45	32/25	8/5
35/18	2/1	28/27	7/6	98/81	56/45	14/9
15/8	27/14	2/1	9/8	7/6	6/5	3/2
5/3	12/7	16/9	2/1	28/27	16/15	4/3
45/28	81/49	12/7	27/14	2/1	36/35	9/7
25/16	45/28	5/3	15/8	35/18	2/1	5/4
5/4	9/7	4/3	3/2	14/9	8/5	1/1

6-37. *Stellated hexanies generated by the prime tetrachord 1/1* a b *4/3. The hexany notes are* a, b, *4/3,* ab, *4*a*/3, and 4*b*/3. The 8 extra notes are (1/1)2=1/1,* a^2, b^2, *16/9,* 3ab/2, 4ab/3, 4a/3b, *and* 4b/3a. *The second stellated hexany is based on number 1 of figure 6-29. Instances of each are based on Archytas's enharmonic. The first is generated by prime tetrachord 1/1 28/27 16/15 4/3. The hexany notes are 28/27, 16/15, 4/3, 448/405, 112/81, and 64/45. The second is based on (1) of 6-20.*

FIRST STELLATED TETRACHORDAL HEXANY

1/1	a	b	a^2	ab	b^2	$4a/3b$	4/3	$4b/3a$	$4a/3$	$4b/3$	$4ab/3$	$3ab/2$	16/9	2/1
1/1	28/27	16/15	784/729	448/405	256/225	35/27	4/3	48/35	112/81	64/45	1792/1215	224/135	16/9	2/1
0	63	112	126	175	223	449	498	547	561	610	673	877	996	1200

SECOND STELLATED TETRACHORDAL HEXANY

1/1	b/a	b^2/a^2	b	b^2/a	b^2	$4/3a$	4/3	$4b/3a$	$4a/3$	$4b/3$	$4b^2/3a$	$3b^2/2a$	16/9	2/1
1/1	36/35	1296/1225	16/15	192/175	256/225	9/7	4/3	48/35	112/81	64/45	256/175	288/175	16/9	2/1
0	49	98	112	161	223	435	498	547	561	610	659	862	996	1200

6-38. (a) *Essential tetrachords of the first stellated hexany. For the sake of clarity, the factor 1 (1/1) has been omitted from 1 · a, 1 · b, 1 · 4/3, etc. The · signs are also deleted. The boldfaced notes in each chord are the starting notes of the prime and inverted tetrachords, 1/1* a b *4/3 and 4/3 4/3*a *4/3*b *1/1.*

PRIME				INVERTED			
1/1	*a*	*b*	4/3	4/3	4/3*a*	4/3*b*	**1/1**
4/3	4*a*/3	4*b*/3	16/9	*ab*	*b*	*a*	**3*ab*/2**
b	*ab*	*b*2	4*b*/3	4*a*/3	4/3	4*a*/3*b*	***a***
a	*a*2	*ab*	4*a*/3	4*b*/3	4*b*/3*a*	4/3	***b***
1/1	*a*	*b*	4/3	4*ab*/3	4*b*/3	4*a*/3	***ab***

to complex 12 of 6-19 when transposed so as to begin on the tone *a*. The stellated form of this hexany is the first of 6-37, while complex 1 of 6-19 yields the second of 6-37. The eight supplementary tones of the first stellated hexany are 1/1, a^2, b^2, 16/9, 4*a*/3*b*, 4*ab*/3, 3*ab*/2, and 4*b*/3*a*. These notes may be deduced by inspection of 6-23, the tetrachordal hexany. The first four extra notes are the squares of the elements of the generator, 1/1, a^2, b^2, and 16/9 (x^2, y^2, z^2, and w^2) from 1/1 *a b* and 4/3. The remaining four notes are the mixed product-quotients needed by the subharmonic faces. These have the form $x{\cdot}y{\cdot}z/w$ (3*ab*/2), $x{\cdot}y{\cdot}w/z$ (4*a*/3*b*), $x{\cdot}z{\cdot}w/y$ (4*b*/3*a*), and $y{\cdot}z{\cdot}w/x$ (4*ab*/3). Two stellated hexanies based on Archytas's enharmonic are shown in 6-37.

The notes of the second type of stellated hexany of 6-30 are derived analogously by replacing *a* in the prime tetrachord with *b/a*. The tetrachord 1/1 28/27 16/15 4/3 in the first type is thus replaced by 1/1 36/35 16/15 4/3.

The essential tetrachords of the first stellated hexany are seen in 6-38, and those of the second may be found by analogy. The component tetrachords of the first stellated hexany derived from Archytas's enharmonic are listed in 6-39. Those of the second kind may be derived by replacing the 28/27 of the first tetrachord with 36/35. The other tetrachordal hexanies of 6-18 also generate stellated hexanies, but their tetrachords are bounded by intervals other than 4/3.

6-39. *Essential tetrachords of the 1/1 28/27 16/15 4/3 stellated hexany.*

PRIME				INVERTED			
1/1	28/27	16/15	4/3	4/3	9/7	5/4	**1/1**
4/3	112/81	64/45	16/9	448/405	16/15	28/27	**224/135**
16/15	448/405	256/225	64/45	112/81	4/3	35/27	**28/27**
28/27	784/729	448/405	112/81	64/45	48/35	4/3	**16/15**
1/1	28/27	16/15	4/3	1792/1215	64/45	112/81	**448/405**

7 Harmonization of tetrachordal scales

SCALES BASED ON tetrachords are found in the musics of a large part of the world. Although much of this music is primarily melodic and heterophonic, this is due neither to the intrinsic nature of tetrachords nor to the scales derived from them. Rather, it is a matter of style and tradition. Many, if not most, tetrachordal scales have harmonic implications even if these implications are contrary to the familiar rules of European tonal harmony.

The melodies of the ancient Greeks were accompanied by more or less independent voices, but polyphony and harmony in their traditional senses appear to have been absent. "A feeling for the triad," however, does appear in the later Greek musical fragments, but this may be a modern and not ancient perception (Winnington-Ingram 1936).

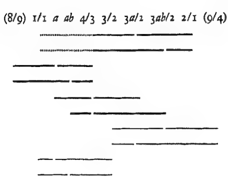
The scales of North Indian music are also based on tetrachords (Sachs 1943; Wilson 1986a, 1987). In this music, drones emphasizing the tonic and usually the dominant of the scale are essential elements of performance. Their function may be to fix the tonic so that ambiguous intervals are not exposed (chapter 5 and Rothenberg 1969, 1978).

Islamic music of the period of the great medieval theorists Al-Farabi, Safiyu-d-Din, and Avicenna (Ibn Sina) was likewise heterophonic rather than harmonic (Sachs 1943; D'Erlanger 1930, 1935, 1938). In recent times, however, some Islamic groups have adopted certain elements of tonal harmony into their music.

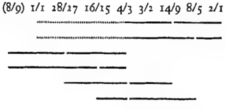
Harmonizing tetrachordal scales

Many tetrachordal scales are nevertheless suitable for harmonic music. The

7-1. *Endogenous harmonization of tetrachordal scales. The addition of the subtonic 9/8 below 1/1 to the enharmonic and chromatic genera where it was called hyperhypate is attested both theoretically and musically (Winnington-Ingram 1936, 25). The dotted lines indicate the lower octave of the dominant of the triads on 4/3.*



7-2. *Endogenous harmonization of Archytas's enharmonic.*



Lydian mode of Ptolemy's intense diatonic genus is the just intonation of the major mode. The diatonic Arabo-Persian scale *hhidjazi*, is more consonant than the 12-tone equal-tempered tuning of the major scale (Helmholtz [1877] 1954).

Harry Partch pointed out that many of the other tetrachordal genera also have harmonic implications which may be exploited in the context of extended just intonation (Partch [1949] 1974). As an example, he offered Wilfrid Perrett's harmonization of a version of the enharmonic tetrachord. Partch added a repeat to Perrett's progression and transposed it into his 43-tone scale (Partch [1949] 1974; Perrett 1926).

Partch also challenged his readers to limit themselves to the notes of the scale. 7-1 depicts the triadic resources of a generalized tetrachordal scale in which both tetrachords are identical. The dark lines delimit triads which are available in all genera while the light ones indicate chords which may or may not be consonant in certain genera.

The three sub-intervals of the tetrachord are denoted as *a*, *b*, and 4/3*ab*, resulting in the tones, 1/1, *a*, *ab*, and 4/3, duplicated on the 3/2. Because there is both musical and literary evidence for the customary addition of the note hyperhypate a 9/8 whole tone below the tonic in the enharmonic and chromatic genera (Winnington-Ingram 1936, 25), it has been included. The inversion of this interval has also been added to allow the construction of a consonant dominant triad in some genera or permutations.

The types of these triads depend upon the tuning of the tetrachord. In Archytas's enharmonic genus, the triads on 4/3 and 8/9 will be septimal minor, 6:7:9. The triad on *a* (28/27) is the septimal major triad, 14:18:21. The triad on *ab* (16/15) is a major triad, 4:5:6, and the alternative triads on 4/3 and 8/9, are minor, 10:12:15. The tonal center appears not to be the 1/1, but rather the 4/3 or mese. These chords are shown in 7-2.

The tonal functions of these triads are determined by the mode or circular permutation of the scale. The Lydian or C mode of Ptolemy's intense diatonic, in its normal form, 16/15 · 9/8 · 10/9, is the familiar major mode with 4:5:6 triads on 1/1, 4/3, and 3/2. The reverse arrangement of this tetrachord, 10/9 · 9/8 · 16/15, generates the natural minor mode with 10:12:15 or subharmonic 4:5:6 triads on these degrees. This scale is not identical to the Hypodorian or A mode of the first scale because that scale has a 27/20 rather than a 4/3 as its fourth degree. The chordal matrices and tetrachordal forms of these scales are shown in 7-3.

7-3. *The 4:5:6 triad and its derived tritriadic scale. The tritriadic or matrix form is the C or Lydian mode of the tetrachordal scale. The tonic of the triad is denoted* t *or 1/1, the third or mediant,* m *and the fifth or dominant,* d. *The tetrachordal form is the E or Dorian mode of the tritriadic scale.*

SUBDOMINANT	4/3 5/3 2/1	*2/d m/d 2/1*
TONIC	1/1 5/4 3/2	*1/1 m d*
DOMINANT	3/2 15/8 9/8	*d d·m d^2*

1/1 9/8 5/4 4/3 3/2 5/3 15/8 2/1
9/8 · 10/9 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15

THE TETRACHORDAL FORM

1/1 16/15 6/5 4/3 3/2 8/5 9/5 2/1
16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9

(16/15 · 9/8 · 10/9)

THE 10:12:15 TRIAD & ITS DERIVED TRITRIADIC SCALE

SUBDOMINANT	4/3 8/5 2/1	*2/d m/d 2/1*
TONIC	1/1 6/5 3/2	*1/1 m d*
DOMINANT	3/2 9/5 9/8	*d d·m d^2*

1/1 9/8 6/5 4/3 3/2 8/5 9/5 2/1
9/8 · 16/15 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9

THE TETRACHORDAL FORM

1/1 10/9 5/4 4/3 3/2 5/3 15/8 2/1
10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 9/8 · 16/5

(10/9 · 9/8 · 16/15)

The seven modes or octave species of the reversed tetrachord scale are the exact inversions of those of the major scale above. The C mode of this scale is the diatonic scale of John Redfield (1928, 191–197). Redfield assigned Hebraic names to these modes and termed the triads with the comma-enlarged fifth "Doric."

The mode that is the inversion of the major scale may be harmonized with three triads built downwards from 2/1, 3/2, and 4/3. An otherwise obscure composer named Blainville wrote a short symphony in this scale and was ridiculed by Rousseau for doing so (Perrett 1931; Partch [1949] 1974). This kind of inverted harmony was called the *phonic system* by the nineteenth and early twentieth century theorist von Öttingen (Helmholtz [1877] 1954; Mandelbaum 1961) in contrast to the traditional *tonic* system.

Tritriadic scales

The scales derived from tetrachords with 9/8 as their second interval may be called *tritriadics* because they may be divided into three triads on the roots 1/1, 4/3, and 3/2. They are harmonizable with analogs of the familiar I IV (I) V I and I IV (VII) III VI (II) V I progressions (Chalmers 1979, 1986, 1987, 1988).

In general, however, the VII and II chords will be out of tune (Lewin 1982) and probably should be omitted in the progressions unless extra notes are employed. The composer Erling Wold, however, has made a case for a more adventurous utilization of available tonal resources (Wold 1988). Partch ([1949] 1974) has done so too in a discussion of a letter from Fox-Strangways concerning the alleged defects of just intonation and their effect on modulation.

The three primary triads on 1/1, 4/3, and 3/2 are of the same type, but the triads on the third (mediant) and sixth (submediant) degrees are of the conjugate or 3/2's complement type. For example, the primary triads of number 1a of 7-4 are major, while the mediant and submediant triads are minor. In number 1b, the modalities are just the reverse. In addition to the principle triads of these scales, triads on other degrees may also be usable. Similarly, in some tunings, seventh or other chords may be useful.

Phonic or descending harmonizations are also possible in certain modes of tritriadic scales. Lewin, in fact, proposes what might be called both phonic major and minor harmonizations (Lewin 1982).

The generalized triad is denoted as *t:m:d*, after Lewin (1982), where *t* is the tonic, *m* the mediant, and *d* the dominant. In principle, any tetrachord containing the interval 9/8 can be arranged as a tritriadic generator, but the majority of the resulting triads will be relatively discordant. If the mediant of a triad is denoted by *m*, then the tetrachord has the form $4/3m \cdot 9/8 \cdot 8m/9$, where $4/3m \cdot 8m/9 = 32/27$. The conjugate tritriadic scale is generated by the permutation $8m/9 \cdot 9/8 \cdot 4/3m$. The magnitude of *m* may range from 9/8 to 4/3 and generate a seven tone tritriadic scale, though the Rothenberg propriety (chapter 5) of the scale and the consonance of the triads will depend of the value of *m*.

Triads with perfect fifths ($d = 3/2$) whose mediants (*m*) are greater than 32/27 and less than 81/64 generate strictly proper scales (chapter 5; Rothenberg 1969, 1975, 1978; Chalmers 1975). Strictly proper scales tend to be perceived as musical gestalts and are used in styles where motivic transposition is an important structural element. Improper scales, on the other hand, are usually employed as sets of principal and auxiliary or ornamental tones.

Only a limited number of acceptably consonant triads exist in just intonation and also generate useful tritriadic scales. The most important of these have been tabulated in 7-4. As indicated above, triads 1a and 1b generate the major and natural minor modes, and 2a and 2b generate the

7-4. *Tritriadic tetrachords. I stands for "improper," and SP for "strictly proper" (Rothenberg 1969, 1975, 1978). In just intonation, tritriadic scales are either strictly proper or improper.*

	TRIAD	MED.	CTS	TETRACHORD	PROPRIETY
1A.	4:5:6	5/4	386	16/15 · 9/8 · 10/9	SP
1B.	10:12:15	6/5	316	10/9 · 9/8 · 16/15	SP
2A.	6:7:9	7/6	267	8/7 · 9/8 · 28/27	I
2B.	14:18:21	9/7	435	28/27 · 9/8 · 8/7	I
3A.	18:22:27	11/9	347	12/11 · 9/8 · 88/81	SP
3B.	22:27:33	27/22	355	88/81 · 9/8 · 12/11	SP
4A.	26:32:39	16/13	359	13/12 · 9/8 · 128/117	SP
4B.	32:39:48	39/32	342	128/117 · 9/8 · 13/12	SP
5A.	22:28:33	14/11	418	22/21 · 9/8 · 112/99	I
5B.	28:33:42	33/28	284	112/99 · 9/8 · 22/21	I
6A.	10:13:15	13/10	454	40/39 · 9/8 · 52/45	I
6B.	26:30:39	15/13	248	52/45 · 9/8 · 40/39	I
7A.	22:26:33	13/11	289	44/39 · 9/8 · 104/99	I
7B.	26:33:39	33/26	413	104/99 · 9/8 · 44/39	I
8A.	14:17:21	17/14	336	56/51 · 9/8 · 68/63	SP
8B.	34:42:51	21/17	366	68/63 · 9/8 · 56/51	SP
9A.	16:19:24	19/16	298	64/57 · 9/8 · 19/18	I
9B.	38:48:57	24/19	404	19/18 · 9/8 · 64/57	I
10A.	64:81:96	81/64	408	256/243 · 9/8 · 9/8	I
10B.	54:64:81	32/27	294	9/8 · 9/8 · 256/243	I
11A.	26:34:39	17/13	464	52/51 · 9/8 · 136/11	I
11B.	34:39:51	39/34	238	136/117 · 9/8 · 52/51	I
12A.	14:16:21	8/7	231	7/6 · 9/8 · 64/63	I
12B.	16:21:24	21/16	471	64/63 · 9/8 · 7/6	I
13A.	20:23:30	23/20	242	80/69 · 9/8 · 46/45	I
13B.	46:60:69	30/23	460	56/45 · 9/8 · 80/69	I
14A.	18:23:27	23/18	424	24/23 · 9/8 · 92/81	I
14B.	46:54:69	27/23	278	92/81 · 9/8 · 24/23	I
15A.	38:46:57	23/19	331	184/171 · 9/8 · 76/69	SP
15B.	46:57:69	57/46	371	76/69 · 9/8 · 184/171	SP

7-5. Mixed tritriadic scales. The triads are 4:5:6 and 6:7:9. (Poole 1850). Mixed scales may often be decomposed into two tetrachords and a disjunctive tone in more than one way. Farnsworth's scale is a mode of Poole's. It may be construed as a tonic major triad, a dominant seventh chord, or a septimal minor triad (6:7:9) on the supertonic (Farnsworth 1958, 1969).

POOLE'S "DOUBLE DIATONIC" OR "DICHORDAL SCALE"

SUBDOMINANT	4/3 5/3 2/1	*2/d x 2/1*
TONIC	1/1 5/4 3/2	*1/1 m d*
DOMINANT	3/2 7/4 9/8	*d s d^2*

1/1 9/8 5/4 4/3 3/2 5/3 7/4 2/1
9/8 · 10/9 · 16/15 · 9/8 · 10/9 · 21/20 · 8/7

ALTERNATE TETRACHORDAL FORM

1/1 10/9 7/6 4/3 3/2 5/3 16/9 2/1
10/9 · 21/20 · 8/7 · 9/8 · 10/9 · 16/15 · 9/8

FARNSWORTH'S SCALE

SUBDOMINANT	21/16 27/16 2/1	*d·s d^3 2/d*
TONIC	1/1 5/4 3/2	*1/1 m d*
DOMINANT	3/2 15/8 9/8 21/16	*d d·m d^2 d·s*

1/1 9/8 5/4 21/16 3/2 27/16 15/8 2/1
9/8 · 10/9 · 21/20 · 8/7 · 9/8 · 10/9 · 16/15

TETRACHORDAL FORM

1/1 9/8 5/4 4/3 3/2 5/3 7/4 2/1
9/8 · 10/9 · 16/15 · 9/8 · 10/9 · 21/20 · 8/7

corresponding septimal minor and septimal major scales. The septimal minor or subminor scale sounds rather soft and mysterious, but the septimal major is surprisingly harsh and discordant. Triads 9a and 9b are virtually equally tempered and sound very much like their 12-tone counterparts. The scales based on 10a and 10b are the Pythagorean tunings of the major and minor modes in which the thirds are the brilliant, if somewhat discordant, 81/64 and 32/27.

Triads with *undecimal, tridecimal,* and *septendecimal* thirds (numbers 3a–8b of 7-4) are less consonant than those discussed above. However, these triads are still relatively smooth and may be useful in certain contexts. Their tetrachords are also interesting melodically as they approximate certain medieval Islamic and neo-Aristoxenian genera (chapter 4). The tetrachords generated by the even less harmonious triads 24:31:36, 64:75:96, 34:40:51, 30:38:45, and 24:29:36 and their conjugates will be found in the Main Catalog.

Scales with mixed triads

Tritriadic scales may also be constructed from triads with different mediants, provided that *d* remains 3/2. An example where the tonic and subdominant triads are 4:5:6 and the dominant triad is 6:7:9 is shown in 7-5 (Helmholtz [1877] 1954, 474). The tetrachordal structure may be described as 9/8 · 8*m*/9 · 4/3*m* (where *m* is the mediant of the tonic triad) for the lower tetrachord and 2*x*/3 · *s*/*x* · 2/*s* (where *x* and *s* are the sixth and seventh of the scale) for the upper tetrachord. However, as 7-5 indicates, mixed tritriadics may often be divided into two tetrachords and a disjunctive tone is more than one way.

Farnsworth's scale, also shown in 7-5, is a mode of Poole's Double Diatonic (Farnsworth 1969). It may be construed as a major triad on 1/1, a dominant seventh chord on 3/2, and a subminor triad (6:7:9) on 9/8.

In chapter 5, the limits on the propriety of mixed modes are discussed.

Ellis's duodenes

Composers may find the intrinsic harmonic resources of tetrachordal scales rather sparse, even with the addition of one or more historically motivated supplementary tones. Two simple remedies immediately come to mind. One is to enlarge the chain of chordal roots of tritriadic scales to encompass four or more triads. This procedure may tend to hide the tetrachords beneath a mass of chords, but by way of compensation,

more tetrachords are created. The process may be seen in 7-6. The parent tritriadic scale contains five tetrachords, all of which are permutations of 16/15 · 9/8 · 10/9 (112 + 204 + 182 cents). The new *pentatriadic* scale contains 42 tetrachords of six different genera.

The second solution is to extend both the *d* and *m* axes to generate structures analogous to A. J. Ellis's *duodenes*, the twelve note "units of modulation" in his theory of just intonation in European tonal harmony (Helmholtz [1877] 1954). The duodene generated from the 4:5:6 triad and some analogs generated by other triads are illustrated in 7-7. These scales likewise consist of large numbers of tetrachords of diverse genera in a harmonic context.

Perrett's harmonizations

Wilfrid Perrett, an English theorist, developed some highly imaginative, if controversial, ideas about Greek music and its early history. In *Some Questions of Musical Theory*, Perrett harmonized a version of the enharmonic tetrachord (21/20 · 64/63 · 5/4) which he attributed to Tartini, but it is more likely that Pachymeres has priority. Perrett used familiar tonic, subdominant, and dominant chord progressions by adding tones, effectively embedding the tetrachord in a larger microchromatic gamut (Perrett 1926, 1928, 1931, 1934). It is this harmonization that Partch quoted in *Genesis of*

7-6. Pentatriadic scales. A pentatriadic is an expansion of a tritriadic by the addition of the subdominant of the subdominant and the dominant of the dominant. An alternative form has a third dominant in place of the second subdominant and is a mode of the scale above.

THE 4:5:6 TRIAD AND A DERIVED PENTATRIADIC SCALE

	16/9 10/9 4/3	$2/d^2$ m/d^2 $2/d$
SUBDOMINANT	4/3 5/3 2/1	$2/d$ m/d $2/1$
TONIC	1/1 5/4 3/2	$1/1$ m d
DOMINANT	3/2 15/8 9/8	d $d{\cdot}m$ d^2
	9/8 45/32 27/16	d^2 $m{\cdot}d^2$ d^3

1/1 10/9 9/8 5/4 4/3 45/32 3/2 5/3 27/16 16/9 15/8 2/1
10/9 · 81/80 · 10/9 · 16/15 · 135/128 · 16/15 · 10/9 · 81/80 · 256/243 · 135/128 · 16/15

TETRACHORDS IN SCALE

RATIOS	CENTS	NUMBER
1. 81/80 · 256/243 · 5/4	22 + 90 + 396	3
2. 256/243 · 135/128 · 6/5	90 + 92 + 316	3
3. 135/128 · 16/15 · 32/27	92 + 112 + 294	8
4. 81/80 · 10/9 · 32/27	22 + 182 + 294	7
5. 16/15 · 9/8 · 10/9	112 + 204 + 182	18
6. 256/243 · 9/8 · 9/8	90 + 204 + 204	3

a Music (Partch [1949] 1974, 171). Perrett placed the tetrachord in the soprano voice and added sufficient extra tones in the lower registers to obtain the desired chord progression. 7-8 simplifies Partch's presentation by leaving out the repeated chords under 16/15, 21/20, and 1/1 that follow the one under 4/3, and by transposing the pitches from 5/3 to 1/1.

Perrett also devised harmonizations for a number of other tetrachords listed by Ptolemy. These harmonizations are shown in 7-9 where they have been transposed to 1/1 and tabulated in a standard format.

Perrett also discovered a harmonization of Archytas's enharmonic, 28/27 · 36/35 · 5/4, a much more plausible and consonant tuning than the 21/20 · 64/63 · 5/4 he chose initially (Perrett 1928, 95). He expressed the solution in the 171-tone equal temperament and later translated it into a

7-7. *Ellis's duodenes. This table is based on Helmholtz [1877] 1954, 457-464. The axes have been reversed from the original in which the chain of 3/2's was vertical. Note the interlocking prime (major) and conjugate (minor) triads. The 4:5:6 duodene contains 54 tetrachords of diverse genera. 10:12:15 is a conjugate duodene which should be compared with the one above of which it is not a "mode." It contains 48 tetrachords of different genera. 6:7:9 is a non-tertian duodene. It contains 62 tetrachords of various genera.*

TRADITIONAL DUODENE BASED ON THE 4:5:6 TRIAD			
5/3	5/4	15/8	45/32
4/3	1/1	3/2	9/8
16/15	8/5	6/5	9/5
DUODENE BASED ON THE 10:12:15 TRIAD			
8/5	6/5	9/5	27/20
4/3	1/1	3/2	9/8
10/9	5/3	5/4	15/8
DUODENE BASED ON THE 6:7:9 TRIAD			
14/9	7/6	7/4	21/16
4/3	1/1	3/2	9/8
8/7	12/7	9/7	27/14

7-8. *Perrett's harmonization of Pachymeres's enharmonic. The numbers under the note ratios represent the harmonic factors or Partch "Identities" of the chords. The uppermost voice contains the tones of the tetrachord. The ratios of each of the chordal components are shown below. Asterisks indicate the roots of harmonic chords, "Otonalities" in Partch's nomenclature. The 28/15 does not occur in the Partch gamut, but a transposed version is available in Partch's system starting on 1/1 = 5/3. The pitches of the tetrachord then become 5/3 7/4 16/9 and 10/9.*

1/1	21/20	16/15	4/3
5	7	8	5
4	6	7	4
3	5	6	3
1	1	1	1
5 = 2/1	7 = 21/20	8 = 16/15	5 = 4/3
4 = 8/5	6 = 9/5	7 = 28/15	4 = 16/15
3 = 6/5	5 = 3/2	6 = 8/5	3 = 8/5
1 = 8/5	1 = 6/5	1 = 16/15	1 = 16/15
8/5 *	6/5 *	16/15 *	16/15 *

17-limit just intonation (Perrett 1934, 158). This harmonization is shown as number 7 of 7-9.

I have devised another harmonization, which is noteworthy in that the movement between the roots of last two chords of the cadence is by a 40/27 rather than a 3/2. This example is shown in 7-10.

These harmonizations are rather simple, with few nonharmonic tones or passing chords. More sophisticated techniques including the use of subharmonic chords would seem appropriate.

More complex treatment is obviously possible in larger microchromatic scales such as Partch's 43-tone gamut. With the help of a computer, 4022 occurrences of tetrachords and 1301 heptatonic scales in which both tetrachords are identical have been found in this scale. Among these are the instances of the *Ptolemaic sequence*, Partch's name for the major mode, and a number of other tetrachords from Ptolemy's catalog. Smaller systems such as Perrett's 19-tone scale have considerable tetrachordal resources; 269 tetrachords and 52 heptatonic tetrachordal scales occur in this gamut.

7-9. Perrett's other tetrachord harmonizations. The names for numbers 3 and 4 are Perrett's; the tetrachord is actually Archytas's diatonic and Ptolemy's tonic diatonic genus rearranged. In ascending form, the tetrachord of numbers 1 and 6 is 28/27 · 15/14 · 6/5, Ptolemy's soft chromatic.

1. INVERTED PTOLEMY'S SOFT CHROMATIC

1/1	6/5	9/7	4/3
5	5	9	7
4	6	7	6
3	4	5	5
1	1	2	1

2. PTOLEMY'S SOFT CHROMATIC

1/1	28/27	10/9	4/3
6	7	5	6
5	6	4	5
4	5	3	4
1	1	1	1

3. PTOLEMY'S "SOFT DIATONIC," REARRANGED

1/1	28/27	7/6	4/3
6	7	7	8
5	6	6	7
4	5	5	6
1	1	1	1

4. PTOLEMY'S "SOFT DIATONIC," REARRANGED, ALTERNATIVE CHORDS

1/1	28/27	7/6	4/3
6	7	5	8
5	6	4	7
4	5	3	6
1	1	1	1

5. ARCHYTAS'S DIATONIC

1/1	28/27	32/27	4/3
6	14	16	16
5	12	12	12
4	9	9	8
2	4	6	5

6. INVERTED PTOLEMY'S SOFT CHROMATIC, ALTERNATIVE CHORDS

1/1	6/5	9/7	4/3
5	5	90	20
4	6	70	15
3	4	63	12
1	1	45	10

7. ARCHYTAS'S ENHARMONIC

1/1	28/27	16/15	4/3
8—16	12	28	6
5—10	10	24	5
3—7	7	17	4
2—4	4	10	

7-10. *Another harmonization of Archytas's enharmonic. The root of the chord under 28/27 is 40/27 a syntonic comma lower than 3/2. The septimal tetrad on 16/15 lacks a major third.*

1/1	28/27	16/15	4/3
5	7	8	5
4	6	7	4
3	5	6	3
1	1	1	1

Many of these tetrachords closely approximate divisions based on higher harmonics or equal temperaments, such as those found in Aristoxenian theory. Because they are composed of secondary or multiple number ratios whose factors are limited to 11, their tones may be harmonized by comparatively simple harmonic or subharmonic chords in a tetradic or hexadic texture.

Wilson's expansions

Perhaps the most innovative technique for harmonizing tetrachords is due to Ervin Wilson (personal communication, 1964). Wilson's technique is based on sequences of chords of increasing intervallic span linked by a common tone. Wilson's have the property that the successive differences between the chordal factors follow a consistent pattern. This pattern is termed the *unit-proportion* (UP). It controls both the rate of intervallic expansion and less directly the degree of consonance. For harmonic chords, it may be expressed as a string of signed, positive integers, i.e., the unit-proportion of the major triad 4:5:6:8 is +1 +1 +2. Subharmonic unit-proportions are written with prefixed – signs; the unit-proportion of the chord 8:6:5:4 is –2 –1 –1. Sequences of chords with identical unit-proportions make up an expansion which progresses from a dense, relatively discordant chord through chords of decreasing tension to a stable consonance, usually a triad with the root doubled.

Sequences of such chords may be used in many musical contexts, and somewhat similar chordal sequences have been explored by Fokker (1966, 1975). Wilson's expansions are particularly attractive when applied to tetrachords and tetrachordal scales.

The application of Wilson's technique to tetrachordal scales is best seen by example. Wilson's original examples were harmonizations of the inverted enharmonic genera, 1/1 5/4 9/7 4/3 (Archytas) and 1/1 5/4 13/10 4/3 (Avicenna) approximated in 22- and 31-tone equal temperament. These examples have been translated into just intonation and are shown in 7-11. An optional 7:8:9:11 chord has been added to Wilson's original progression for the inverted Archytas's enharmonic.

Although one may limit the harmonization to a single tetrachord, it is more likely that one will want to harmonize all seven tones of the scale. Several solutions to this rather difficult problem using both harmonic and subharmonic chords with varied unit-proportions and different common tones are given in 7-12. In these examples, either the 4/3 or 3/2 is held

constant throughout the progression. A passing chord containing intervals of 13 and 15 is used in number 2 to make the progression smoother. These intervals are conditioned in part by the unit-proportion of the set and in part by the intervals of the tetrachord. The major caveat is to limit the number of chords and extra tones when preservation of the melody of the tetrachord is important.

Except for octave transposition of some of the chordal tones and occasional passing chords there has not been much study of harmonic elaboration (Wilson, personal communication). This is true of the endogenous and tritriadic approaches as well. The standard techniques, however, would appear to be applicable here as in traditional practice, but only more experimentation will tell.

Although the majority of this chapter has been presented from the viewpoint of just intonation, these scales and their various harmonizations are equally valid in systems of equal temperament which furnish adequate approximations to the important melodic and harmonic intervals.

7-11. *Wilson's expansion technique. The set of ratios are the chordal tones relative to 1/1. (1) is the just intonation version of Wilson's first expansion harmonization with the later addition of an optional 7 8 9 11 chord at the beginning. The original was quantized to 22-tone equal temperament. (2) is the just intonation version of Wilson's second expansion harmonization. The original was quantized to 31-tone equal temperament. In both cases, the added tones are in lighter type. The optional chord is in parentheses.*

1. INVERTED ARCHYTAS ENHARMONIC, HARMONIC CHORDS ON 3/2, UP = +1 +1 +2

1/1			5/4	9/7		4/3	3/2		15/8	27/14	2/1
		(7				8	9	11)			
		(7/6				4/3	3/2	11/6)			
		6			7		8		10		
		9/8			21/16		3/2		15/8		
	5			6			7			9	
	15/14			9/7			3/2			27/14	
4			5				6				8
1/1			5/4				3/2				2/1

2. INVERTED AVICENNA'S ENHARMONIC, HARMONIC CHORDS ON 3/2, UP = +3 +3 +6

1/1			5/4		13/10		4/3	3/2	15/8	39/20	2/1
		18				21		24	30		
		9/8				21/16		3/2	15/8		
	14			17				20		26	
	21/20			51/40				3/2		39/20	
12			15					18			24
1/1			5/4					3/2			2/1

7-12. *Trial expansion harmonizations. The successive differences or unit proportions are positive in harmonic chords, negative in subharmonic. The non-scalar added tones are in lighter type. Passing notes are in parentheses.*

1. DIDYMOS'S CHROMATIC, SUBHARMONIC CHORDS ON 4/3, UP = −5 −3 −2

1/1 16/15 10/9 4/3 3/2 8/5 5/3 2/1

30	25	22	20
10/9	4/3	50/33	5/3

25	20	17	15
16/15	4/3	80/51	16/9

20	15	12	10
1/1	4/3	5/3	2/1

2. HARMONIC CHORDS, 3/2 COMMON, PASSING NOTES INSERTED, UP = +1 +2 +3

1/1 7/6 5/4 4/3 3/2 7/4 15/8 2/1

15	16	18	21
5/4	4/3	3/2	7/4

(12)	(13)	15	(18)
(6/5)	(13/10)	3/2	(9/5)

9	10	12	15
9/8	5/4	3/2	15/8

6	7	9	12
1/1	7/6	3/2	2/1

3. ARCHYTAS'S ENHARMONIC, SUBHARMONIC CHORDS ON 4/3, UP = +2 −1 −1

1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1

11	9	8	7
12/11	4/3	3/2	12/7

10	8	7	6
16/15	4/3	32/21	16/9

9	7	6	5
28/27	4/3	14/9	28/15

8	6	5	4
1/1	4/3	8/5	2/1

4. INVERTED DIDYMOS'S CHROMATIC, HARMONIC CHORDS ON 3/2, UP = +2 +3 +5

1/1 6/5 5/4 4/3 3/2 9/5 15/8 2/1

20	22	25	30
6/5	33/25	3/2	9/5

15	17	20	25
9/8	51/40	3/2	15/8

10	12	15	20
1/1	6/5	3/2	2/1

5. ARCHYTAS'S ENHARMONIC, 4/3 COMMON, HARMONIC CHORDS, UP = +2 +2 +2

1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1

14	16	18	20
7/6	4/3	3/2	5/3

10	12	14	16
10/9	4/3	14/9	16/9

8	10	12	14
16/15	4/3	8/5	28/15

7	9	11	13
28/27	4/3	44/27	52/27

6	8	10	12
1/1	4/3	5/3	2/1

6. INVERTED ARCHYTAS'S ENHARMONIC, SUBHARMONIC CHORDS ON 3/2, UP = −2 −2 −2

1/1 5/4 9/7 4/3 3/2 15/8 27/14 2/1

20	18	16	14
6/5	4/3	3/2	12/7

16	14	12	10
15/14	5/4	3/2	15/8

14	12	10	8
27/26	27/22	3/2	27/14

13	11	9	7
9/8	9/7	3/2	9/5

12	10	8	6
1/1	6/5	3/2	2/1

8 Schlesinger's harmoniai, Wilson's diaphonic cycles, and other similar constructs

THE HARMONIAI WERE proposed by the English musicologist Kathleen Schlesinger as a reconstruction and rediscovery of the original forms of the modal scales of classical Greek music. Schlesinger spent many years developing her theories by experimenting with facsimiles of ancient auloi found in archaeological sites in Egypt, Pompeii, and elsewhere. Later, she extended her studies to include flutes of ancient and modern folk cultures. As a result of her researches, she questioned the accepted interpretation of Greek musical notation. The results of these studies were previewed in a paper on Aristoxenus and Greek musical intervals (Schlesinger 1933) and were presented at length in her major work, *The Greek Aulos* (1939). Her writings are a major challenge to the traditional tetrachord-based doctrines of the Aristoxenian and Ptolemaic theorists. While there are compelling reasons to doubt that her scales were ever a part of Greek musical practice, they form a musical system of great ingenuity and potential utility in their own right.

This first part of this chapter is devoted to an exposition and analysis of her work. Various extensions and additions are proposed and near the end related materials, including Wilson's diaphonic cycles, are discussed.

The Schlesinger harmoniai

Schlesinger's harmoniai are 7-tone sections of the subharmonic series between members an octave apart. In theory, they are generated by aliquot divisions of the vibrating air columns of wind instruments. The same intervals, however, are obtained by the linear division of half strings. As string lengths are conceptually simpler than air columns, this discussion

8-1. *The diatonic Perfect Immutable System in the Dorian tonos according to Schlesinger. Each diatonic harmonia may be taken as an octave species of this system. (As elsewhere, at variance from Schlesinger, hypate meson is equated with E rather than F.) Trite synemmenon is required for the hypo-modes, in which it replaces paramese. The diatonic synemmenon tetrachord consists of the numbers 16 15 13 and 12.*

NOTE	M.D.	TRANS.
PROSLAMBANOMENOS	32	A
HYPATE HYPATON	28	B
PARHYPATE HYPATON	26	C
LICHANOS HYPATON	24	D
HYPATE MESON	22	E
PARHYPATE MESON	20	F
LICHANOS MESON	18	G
MESE	16	a
TRITE SYNEMMENON	15	b♭
PARAMESE	14	b
TRITE DIEZEUGMENON	13	c
PARANETE DIEZEUGMENON	12	d
NETE DIEZEUGMENON	11	e
TRITE HYPERBOLAION	10	f
PARANETE HYPERBOLAION	9	g
NETE HYPERBOLAION	8	a'

will refer to the former for clarity. The numbers or *modal determinants* assigned to each of the notes are to be understood as the denominators of ratios. The sequence 22 20 18 16 is a shorthand for the notes 22/22 22/20 22/18 22/16 or 1/1 11/10 11/9 11/8 above the tonic note 22.

The octave rather than the tetrachord is the fundamental module of these scales. Although the scales can be analyzed into tetrachords and disjunctive tones, the tetrachords are of different sizes which, in general, do not equal 4/3. Furthermore, each interval of the scale is different; the series of duplicated conjunct and disjunct tetrachords of the traditional theorists (chapter 6) is replaced by modal heptachords which repeat only at the octave.

The familiar names for the octave species are retained, but each modal octave is, in effect, another segment of the subharmonic series, bounded by a different modal determinant and its octave. 8-1 shows the form the Perfect Immutable System in the diatonic genus takes in her theory.

The modal determinants have many of the functions of tonics. As such, they serve to identify and define the harmoniai. Schlesinger also considers that mese itself has tonic functions, a point which is controversial even in the standard theory (Winnington-Ingram 1936).

The relations the other octave species have to the central Dorian octave is shown in 8-2. The seven harmoniai may also be constructed on a common tone, proslambanomenos, by assigning their modal determinants to hypate meson. In this case, there are six additional keys or tonoi which are named after the homonymous harmoniai. The Dorian and the other modal octaves are then found at corresponding transpositional levels in each tonos. Con-

8-2. *The diatonic harmoniai as octave species of the Perfect Immutable System in the Dorian tonos. Other tonoi are defined by assigning their modal determinants to hypate meson and proceeding through the subharmonic series. The Dorian, however, is the basis for Schlesinger's theory.*

	PS	HH	PH	LH	HM	PM	LM	M	TS	PM	TD	PD	ND	TH	PN	NH
	32	28	26	24	22	20	18	16	15	14	13	12	11	10	9	8
	A	B	C	D	E	F	G	a	b♭	b	c	d	e	f	g	a'
MIXOLYDIAN		28	26	24	22	20	18	16		14						
LYDIAN			26	24	22	20	18	16		14	13					
PHRYGIAN				24	22	20	18	16		14	13	12				
DORIAN					22	20	18	16		14	13	12	11			
HYPOLYDIAN						20	18	16	(15)	14	13	12	11	10		
HYPOPHRYGIAN							18	16	15		13	12	11	10	9	
HYPODORIAN								16	15		13	12	11	10	9	8

comitantly, there is a seven-fold differentiation of the tuning of the other notes of the Perfect Immutable System. These tonoi are shown in 8-3.

Anomalies and inconsistencies

The clarity and consistency of Schlesinger's system, however, is only apparent. Once one goes beyond the seven diatonic harmoniai, anomalies of various types soon appear.

Schlesinger explicitly denies harmonia status to the octave species running from proslambanomenos to mese, calling it the *bastard Hypodorian* or *Mixophrygian*. She rejects it because it resembles the Hypodorian an octave lower but differs in having 8/7 rather than 16/15 as its first interval. Yet this scale had a name (Hypermixolydian) in the standard theory and was rejected by Ptolemy precisely because it was merely the Hypodorian transposed by an octave.

Each of the diatonic harmoniai also had chromatic and enharmonic forms derived by subdividing the the first interval of each tetrachord and deleting the former mesopyknon. This process is identified with katapyknosis and is analogous to the derivation of the genera in the standard theory (see chapters 2 and 4). These forms are listed in 8-4 for the central octave of the Perfect Immutable System in each homonymous tonos.

It is also here that some of the most serious problems with her theory occur. Although all of the diatonic harmoniai occur as octave species of the Dorian, and of each other, the chromatic and enharmonic forms of the other harmoniai are not modes of the corresponding forms of the Dorian harmonia. Rather, they are derived by katapyknosis of the homonymous tonos. The symmetry is broken and the modes are no longer identical in

8-3. *Schlesinger's diatonic harmoniai as tonoi. Elsewhere she gives different forms, most notably variants of the Lydian, with 27 instread of 26, and Dorian, with 21 instead of 22 (Schlesinger 1939, 1–35, 142). A trite synemmenon could be defined in each tonos, but Schlesinger chose not to do so. Schlesinger conceived of the Hypolydian harmonia in two forms with 15 alternating with 14 (ibid., 26–27). Her theory demands that the Dorian trite synemmenon (15) be employed in all the hypo-modes, but she allows the alternation in the Hypolydian harmonia.*

	PS	HH	PH	LH	HM	PM	LM	M	PM	TD	PD	ND	TH	PH	NH
	A	B	C	D	E	F	G	a	b	c	d	e'	f'	g'	a'
MIXOLYDIAN	44	40	36	32	28	26	24	22	20	18	16	14	13	12	11
LYDIAN	40	36	32	28	26	24	22	20	18	16	14	13	12	11	10
PHRYGIAN	36	32	28	26	24	22	20	18	16	14	13	12	11	10	9
DORIAN	32	28	26	24	22	20	18	16	14	13	12	11	10	9	8
HYPOLYDIAN	28	26	24	22	20	18	16	15	13	12	11	10	9	8	7
HYPOPHRYGIAN	26	24	22	20	18	16	15	13	12	11	10	9	8	7	13/2
HYPODORIAN	24	22	20	18	16	15	13	12	11	10	9	8	7	13/2	6

different tonoi. Even the modal determinants of the harmoniai may be changed in different tonoi.

Other inconsistencies and anomalies may be noted. The chromatic and enharmonic forms are incompletely separated since the enharmonic and chromatic forms of some harmoniai share tetrachords. Even these presumed canonical forms do not agree with the varieties she derives elsewhere in *The Greek Aulos* from her interpretation of the Greek notation.

Because of certain irregularities in the notation, she claims that the modal determinant of the Lydian harmonia must have been altered at some period from 26 (13) to 27 and that of the Dorian from 22 to 21. These changes of modal determinants would not only have disrupted the tonal relations of the original harmoniai, but would also have affected the tonality of the rest of the system in all three genera. Since the Dorian harmonia was the center of the system, this would not have been a trivial change.

The question of modal determinant 15

Another problem is the status of 15 as a modal determinant. Schlesinger strongly denies the existence of a harmonia whose modal determinant is 15. Yet one of her facsimile instruments plays it easily. She also states that hypate hypaton could be tuned to 30 in the Hypodorian harmonia where it generates a perfectly good harmonia of modal determinant 15 with the octave at trite synemmenon (8-2).

The inclusion of modal determinant 15 is, on the whole, quite problematical. It enters originally as the Dorian trite synemmenon (B♭), the only accidental in the Greater Perfect System. Although Schlesinger mentions what she calls the conjunct Dorian harmonia where 15 substitutes for 14, and elsewhere allows 15 to freely alternate with 14, she uses trite syn-

8-4. *Schlesinger's chromatic and enharmonic harmoniai (Schlesinger 1939, 214). It is clear that these scales are not simply modes of the Dorian chromatic and enharmonic genera, but are derived from the homonymous tonoi. The chromatic and enharmonic forms are derived by two successive doublings of the modal determinant followed by note selection to obtain the desired melodic contours. The upper tetrachords of the chromatic and enharmonic forms of the Dorian and Hypolydian harmoniai are identical. In the Hypolydian harmonia 30 (15) may replace 28 (14). The Hypophrygian and Hypodorian harmoniai have a single enharmonic-chromatic form.*

HARMONIA	CHROMATIC	ENHARMONIC
MIXOLYDIAN	28 27 26 22 20 19 18 14	56 55 54 44 40 39 38 28
LYDIAN	26 25 24 20 18 17 16 13	52 51 50 40 36 35 34 26
PHRYGIAN	24 23 22 18 16 15 14 12	48 47 46 36 32 31 30 24
DORIAN	44 42 40 32 28 27 26 22	44 43 42 32 28 27 26 22
HYPOLYDIAN	40 38 36 28 26 25 24 20	40 39 38 28 26 25 24 20
HYPOPHRYGIAN	36 35 34 26 24 23 22 18	36 35 34 26 24 23 22 18
HYPODORIAN	32 31 30 24 22 21 20 16	32 31 30 24 22 21 20 16

emmenon mainly to construct the diatonic hypo-modes. This is very much at variance with the usage of this note by the standard theorists whose Hypodorian, Hypophrygian, and Hypolydian modes employ only the natural notes of Greater Perfect System.

For these theorists, trite synemmenon and the rest of the synemmenon tetrachord are part of the Lesser Perfect System and are used to primarily illustrate the melodic effect of modulations to the key a perfect fourth lower. Bacchios also employs it to illustrate certain rare intervals such as the ekbole, spondeiasmos, and eklysis (chapters 6 and 7). The combination of the Greater and Lesser Perfect Systems to form the Perfect Immutable System is basically a pedagogical device, not a reflection of musical practice. Furthermore, the Lesser Perfect System terminates with the synemmenon tetrachord, but to complete Schlesinger's hypo-harmoniai the note sequence would have to switch back into the notes of the Greater Perfect System. Although chromaticism and modulation occur both in theory and in the surviving fragments (Winnington-Ingram 1936), this use of synemmenon would seem to be most unusual.

Historical evidence

Much of Schlesinger's case for the harmoniai is based on fragmentary quotations from classical Greek writers. This evidence is dubious support at best.

Theorists such as Aristoxenos complain about the unstable pitch and indeterminate tuning of the aulos (Schlesinger 1939). Aristoxenos claims that the intervals of music are determined by the performance skill of the player on both stringed and blown instruments and not by the instruments themselves. This polemic may be interpreted either as referring to the inherent pitch instability of the instrument or to the difficulty of bending the pitches so as to approximate a scale system for which it is not physically suited, i.e. the standard tetrachordal theory. Whatever the correct interpretation, the passage does suggest that Schlesinger's harmoniai played little or no role in Greek musical practice in the fourth century BCE.

The problem lies with our ignorance of the Greek music and its mode of performance. It is quite possible for an instrument to be musically prominent and at the same time difficult to play in acceptable tune. Schlesinger may well have been right about the natural scales of auloi and still be entirely wrong about their employment in Greek music of any period.

The harmoniai in world music

Schlesinger also tries to bolster her argument by appealing to ethnomusicology. Her case for the employment of the harmoniai in non-European folk and art music gives the impression of overpleading, especially in her analysis of Indonesian tunings. It is true, however, that wind instruments from many cultures often have roughly equidistant, equal sized finger holes. For example, the scales of many Andean flutes do appear to resemble sequences of tones from the various harmoniai, although the scales may not be identical throughout the gamut (Ervin Wilson, personal communication). The scales on these instruments are usually pentatonic, rather than heptatonic. Often one or more tones will diverge from the heptatonic pattern, particularly with respect to the vent, which is tuned to bring out the pentatonic structure. Nevertheless, some of the harmoniai sound very similar to the scales heard on recordings of Bolivian and Peruvian music. Hence, these data may serve as at least a partial vindication of her ideas.

Empirical studies on instruments

In *The Greek Aulos*, Schlesinger made use of a large body of data obtained by constructing and playing facsimiles of ancient auloi. She also studied fipple flutes and other folk wind instruments. These studies deserve critical attention.

The chief difficulty one has in evaluating this work is its lack of replication by other investigators. However, there are two published experimental studies which are relevant to her hypotheses.

The first is that of Letter, who made the assumption that two of the holes on the surviving auloi were 4/3 or 2/1 apart (Letter 1969). From measurements on these instruments, he determined the probable reed lengths. His measurements and calculations yielded a number of known tetrachords, including 12/11 · 11/10 · 10/9, 9/8 · 88/81 · 12/11, 9/8 · 16/15 · 10/9, 14/13 · 8/7 · 13/12, and some pentachordal sequences, but little convincing evidence for the subharmonic series or the harmoniai.

More recently, Amos built modal flutes with holes spaced at increments of one-eighth the distance from the fipple to the open end and the studied the resulting intervals (Amos 1981). This procedure, however, is not really in accord with Schlesinger's work. She employed rather complex formulae involving corrections for the diameter and certain other physical parameters to determine the spacing of the holes of modal flutes.

The pitches of Amos's flutes were measured by audibly comparing the flute tone to a calibrated digital oscillator and minimizing beats. Amos's results show that the resulting intervals are subject to wide variation from flute to flute and depend upon humidity, wind pressure, fingering, and other parameters.

While not strictly comparable to Schlesinger's results, the results of these investigators suggest that one should be cautious in extrapolating the tuning of musical systems from the holes of wind instruments.

Schlesinger herself made the same caveat and stated that the aulos alone gave birth to the harmoniai. She claimed that the acoustical properties of the aulos are simpler than those of the flute, and therefore, one can accurately deduce the musical system from the spacing of the finger holes of auloi. People who have made and played aulos-like instruments are less certain.

Lou Harrison found the traditional Korean oboe, the piri (and the homemade miguk piri), to be difficult to play in tune and noted its tendency to overblow at the twelfth (personal communication). Jim French, who has spent a number of years researching the aulos from both an archaeological and an experimental perspective, has discovered that the type of reed and its processing are far more crucial than Schlesinger implies. His results with double auloi indicate that the selection of a particular reed can change the fundamental by a 4/3 (personal communication). Duplicated tetrachords are thus quite natural on this kind of instrument. He has also found that sequences of consecutive intervals from harmoniai such as that on 16 (Hypodorian) are relatively easy to play on these instruments and may be embodied in historical examples and artistic depictions.

Composition with the harmoniai

The question of whether or not Schlesinger's harmoniai are relevant to Greek or world music may be of less importance to the experimental musician than their possible use in composition. Her most fruitful contribution ultimately may be her suggestion that the harmonia be considered a "new language of music" (Schlesinger 1939).

Schlesinger tuned her piano to the Dorian harmonia in which C (at 256 Hertz) equals the modal determinant 22. Thus she used only an 11–pitch gamut. For some unstated reason, she did not give a tuning for the note B♭, which would have had the modal determinant 25, though she did include

such prime numbers as 17 and 19 and composites of comparable size such as 22 and 24. One would think that the Phrygian harmonia on 24 would make more efficient use of the keyboard, unless there are problems with the altered tension of the piano strings. This, of course, would not be a limitation with electronic instruments.

Schlesinger was fortunately able to enlist the composer Elsie Hamilton from South Australia in these efforts. Hamilton composed a number of works in the Dorian diatonic tuning between 1916 and 1929. In 1935, Hamilton trained a chamber orchestra in Stuttgart to perform in the harmoniai. Although several orchestral and dramatic works were composed and performed during this period, it has been impossible to find further information about the composer or discover whether the scores are still extant.

From the excerpts in *The Greek Aulos*, it would appear that Hamilton employed a conservative melodic idiom with straightforward rhythms (8-6). Schlesinger comments that such a simplification was necessary for both "executant and listener." The quotations from the score of *Agave*, brief as they are, seem quite convincing musically in a realization on a retunable synthesizer.

Hamilton's harmonic system is of considerable interest. Although familiar chords are scarce in this system, virtually any interval larger than a melodic second is at least a quasi-consonance. Rather than attempt a translation of tertian harmonic concepts to this tuning, Hamilton instead chose to use the tetrachordal frameworks of the modes as the basic consonances (8-5 and 8-6a). In the Dorian mode, this chord would be 22 16 14 11 (1/1 11/8 11/7 2/1), with 15 (22/15) as an alternative tone.

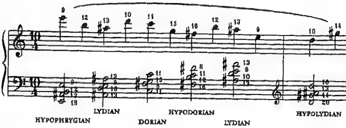
A melodic line may be supported by a succession of such chords taken from all seven of the modes. Hamilton augmented this somewhat sparse

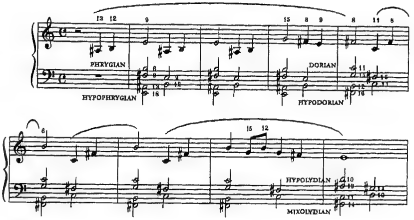
8-5. *Harmonization of Schlesinger's harmoniai. Tetrachordal framework chords. Chords from the "conjunct" harmoniai in which 15 replaces 14 are also shown where applicable.*

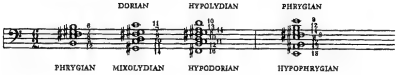
	DISJUNCT	CONJUNCT
MIXOLYDIAN	28:22:20:14	28:22:16:14
LYDIAN	26:20:18:13	26:20:14:13, 26:20:15:13
PHRYGIAN	24:18:16:12	24:18:13:12
DORIAN	22:16:14:11, 22:16:15:11	22:16:12:11
HYPOLYDIAN	20:15:13:10, 20:14:13:10	20:15:11:10, 20:14:11:10
HYPOPHRYGIAN	18:13:12:9	18:13:10:9
HYPODORIAN	16:12:11:8	16:12:9:8

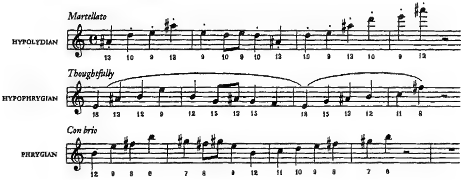
8-6. *Excerpts from* Agave *by Elsie Hamilton, with ratio numbers.*

(a) Tetrachordal framework chords ("Sunrise").



(b) Mixed chorus and tetrachords of resolution ("Funeral March").

(c) Combined framework chords ("Sunrise").



(d) Modal tranposition.

vocabulary with chords formed by the union and intersection of chords from two related harmoniai (8-6b, 8-6c, and 8-7). In the latter case, the chords are resolved to their common dyad.

She also discovered that parallel transposition results in changes of modality which are musically exploitable (8-6d), although the given examples are stated to have been approximated to the piano intonation.

One would characterize her harmonic techniques as essentially polytonal and polymodal, rather than "diatonic" or "chromatic."

8-7. *Chordal relations between related harmoniai (Schlesinger 1939, 543–44).*

D	ML	HL	L	HP	P	HD	D	ML
TETRACHORDAL CHORDS								
11	7	10	13	9	6	8	11	7
7	10	13	9	12	8	11	7	10
8	11	14	10	13	9	12	8	11
11	14	20	13	18	12	16	11	14
MIXED CHORDS								
7	10	13	9	6	8	11	7	
10	13	9	12	8	11	7	10	
8	11	14	10	13	9	12	8	
11	14	20	13	18	12	16	11	
INTERVALS OF RESOLUTION								
11	7	10	13	9	6	8	11	
14	10	13	9	12	8	11	14	

It is a pity that more examples of Hamilton's use of the harmoniai are not extant. From this limited sample, it appears that Schlesinger's system succeeds as a "new language of music."

Schlesinger's harmoniai have inspired other composers, including Harry Partch and Cris Forster. Partch devoted a large part of his chapter on other systems of just intonation to her work, citing it as a justification to proceed on to ratios of 13 (Partch [1949] 1974). He correctly identified her harmoniai with his Utonalities, with the addition of the Secondary Ratio, 16/15. Forster has constructed several instruments embodying the ratios of 13 in a Partch tonality diamond context. He has also composed a considerable body of music for these instruments (Forster 1979).

Extensions to Schlesinger's system

Although Schlesinger's system suffers from internal inconsistencies and omissions, her scales form a fascinating system in their own right, independent of their questionable historical status. The most obvious of the corrections or enhancements is to rationalize her enharmonic and chromatic forms so that all three forms of each harmonia are distinct. The next step is the definition of local tritai synemmenon in each of the tonoi so that correct hypo-modes and conjunct harmoniai may be constructed. Finally, new harmoniai based on modal determinants not used by Schlesinger are proposed. These new modal determinants range from 15 to 33.

Rationalization of the harmoniai

The first and most obvious extension to Schlesinger's system is to furnish distinct chromatic and enharmonic forms for her diatonic harmoniai. This may be done by katapyknosis of the diatonic with the multipliers 2 and 4.

To obtain the corrected chromatic versions, the first interval of each tetrachord of the diatonic harmoniai is linearly divided into two parts. The two new intervals are retained while simultaneously deleting the topmost

note of each tetrachord to create the characteristic interval of the genus. By this process, the old diatonic first intervals become the pykna of the new chromatic forms.

The enharmonic is created analogously by katapyknosis with four. The first two new intervals are retained, leading to pykna which consist of the chromatic first intervals. This procedure is equivalent to performing katapyknosis with two on the chromatic genera resulting from the operations above.

Wilson has suggested performing katapyknosis with 3 to produce *trichromatic* forms (personal communication). Ptolemy used the same technique to generate his shades. This operation produces two forms, a 1 + 1 form in which the two lowest successive intervals are retained and a 1 + 2 form in which the lowest and the sum of the two highest are used. The pykna of the 1 + 1 and 1 + 2 forms are thus different and the 1 + 1 form tends to melodically approximate the enharmonic. A third form, the 2 + 1, potentially exists, but would violate Greek melodic canons (chapter 3).

In an analogous manner, katapyknosis by 5 and 6 are possible if the interval to be divided is large enough. These divisors generate what may be called *pentachromatic*, *pentenharmonic*, *hexachromatic*, and *hexenharmonic* genera. The forms of the rationalized harmoniai including the two trichromatic as well as the pentachromatic genera, created from a 2 + 3 division of the pyknon, are shown in 8-8.

If one generates all the forms of a harmonia which do not violate accepted melodic canons by katapyknosis with the numbers 1 through 6, nineteen genera result. The Hypermixolydian or "bastard Hypodorian" provides a good example of this process because the first diatonic interval is the comparatively large septimal tone 8/7 (231 cents). The nineteen katapyknotic genera of her "bastard Hypodorian" are shown in 8-9.

Local tritai synemmenon

Although all of the diatonic harmoniai can be represented as octave species of the Dorian harmonia (plus trite synemmenon) by choosing different notes as modal determinants, in the homonymous tonoi the central octave is occupied by the notes of the corresponding harmoniai. Since all of the tonoi are structurally as well as logically equivalent, the argument which demanded that 15 replace 14 in the hypo-modes of the Dorian requires that a local trite synemmenon be defined in each tonos. Otherwise, the

8-8. *Rationalized harmoniai. These harmoniai should be compared to Schlesinger's own as significant differences exist between these and some of hers in the chromatic and enharmonic genera. Three new genera are also provided; these are based on katapyknosis by 3 and 5 instead of 2 and 4. To avoid fractions, some numbers have been doubled. In principle, 14 may be substituted for 15 in the hypo-modes. 14 alternates with 15 in the Hypolydian. To preserve melodic contour, the chromatic and enharmonic forms of the Hypodorian are derived from the "bastard" harmonia. The forms of the lower tetrachords of Schlesinger's preferred harmonia would be 32 31 30 24, 48 47 46 36, 48 47 45 36, and 80 78 75 60..*

Mixolydian

DIATONIC
14 13 12 11 10 9 8 7
CHROMATIC
28 27 26 22 20 19 18 14
TRICHROMATIC 1
42 41 40 33 30 29 28 21
TRICHROMATIC 2
42 41 39 33 30 29 27 21
ENHARMONIC
56 55 54 44 40 39 38 28
PENTACHROMATIC
70 68 65 55 50 48 45 35

Lydian

DIATONIC
13 12 11 10 9 8 7 13
CHROMATIC
26 25 24 20 18 17 16 13
TRICHROMATIC 1
39 38 37 30 27 26 25 39
TRICHROMATIC 2
39 38 36 30 27 26 24 39
ENHARMONIC
52 51 50 40 36 35 34 26
PENTACHROMATIC
65 63 60 50 45 43 40 65

Phrygian

DIATONIC
12 11 10 9 8 7 13 6
CHROMATIC
24 23 22 18 16 15 14 12
TRICHROMATIC 1
36 35 34 27 24 23 22 18
TRICHROMATIC 2
36 35 33 27 24 23 21 18
ENHARMONIC
48 47 46 36 32 31 30 24
PENTACHROMATIC
60 58 55 45 40 38 35 30

Dorian

DIATONIC
11 10 9 8 7 13 6 11
CHROMATIC
22 21 20 16 14 27 13 11
TRICHROMATIC 1
33 32 31 24 21 41 40 33
TRICHROMATIC 2
33 32 30 24 21 20 39 33
ENHARMONIC
44 43 42 32 28 55 27 22
PENTACHROMATIC
55 53 50 40 35 34 65 55

Hypolydian

DIATONIC
10 9 8 7 13 6 11 5
CHROMATIC
20 19 18 14 13 25 12 10
TRICHROMATIC 1
30 29 28 21 39 38 37 15
TRICHROMATIC 2
30 29 27 21 39 38 36 15
ENHARMONIC
40 39 38 28 26 51 25 20
PENTACHROMATIC
50 48 45 35 65 63 30 25

Hypophrygian

DIATONIC
18 16 15 13 12 11 10 9
CHROMATIC
18 17 16 13 12 23 11 9
TRICHROMATIC 1
54 52 50 39 36 35 34 27
TRICHROMATIC 2
54 52 48 39 36 35 33 27
ENHARMONIC
36 35 34 26 24 47 23 18
PENTACHROMATIC
90 86 80 65 60 58 55 45

Hypodorian

DIATONIC
16 15 13 12 11 10 9 8
CHROMATIC
32 30 28 24 22 21 20 16
TRICHROMATIC 1
48 46 44 36 33 32 31 24
TRICHROMATIC 2
48 46 42 36 33 32 30 24
ENHARMONIC
64 62 60 48 44 43 42 32
PENTACHROMATIC
80 76 70 60 55 53 50 40

three hypo-modes in each tonos would be merely cyclic permutations of the original sequence and would therefore lack modal distinction. These tritai synemmenon are also needed to to form what Schlesinger would probably term conjunct harmoniai.

The new tritai synemmenon may be supplied by analogy through katapyknosis of the disjunctive tone by 2. These additions, of course, increase the number of possible scale forms, as the new notes may alternate with the lesser of their neighbors as 15 alternates with 14 in the Dorian prototype. This alternation generates fairly wide intervals in the range of augmented seconds and gives the harmoniai containing them a chromatic or harmonic minor flavor not present in the corresponding modes of the Dorian harmonia.

8-9. *The nineteen genera of Schlesinger's "bastard Hypodorian" harmonia. Beyond 6x the intervals are usually too small to be useful melodically. The numbers after the genus abbreviations distinguish the various species. The multiplier refers to the multiplication of the modal determinants in katapyknosis. The species are defined by the unit-proportions of their pykna. The 4x, 5x, and 6x divisions define genera with both enharmonic and chromatic melodic properties.*

NO.	DIVISION	MULTIPLIER	SPECIES
	DIATONIC		
D1	16 14 13 12 11 10 9 8	1x	1+1
	CHROMATIC		
C1	16 15 14 12 11 21 10 8	2x	1+1
	TRICHROMATIC		
T1	24 23 22 18 33 32 31 12	3x	1+1
T2	24 23 21 18 33 32 30 12	3x	1+2
	ENHARMONIC/CHROMATIC		
E1	32 31 30 24 22 43 21 16	4x	1+1
E2	32 31 29 24 22 43 41 16	4x	1+2
E3	32 31 28 24 22 43 20 16	4x	1+3
	PENTACHROMATIC/PENTENHARMONIC		
P1	40 39 38 30 55 27 53 20	5x	1+1
P2	40 39 37 30 55 27 26 20	5x	1+2
P3	40 39 36 30 55 27 51 20	5x	1+3
P4	40 39 35 30 55 27 25 20	5x	1+4
P5	40 38 36 30 55 53 51 20	5x	2+2
P6	40 38 35 30 55 53 50 20	5x	2+3
	HEXACHROMATIC/HEXENHARMONIC		
H1	48 47 46 36 33 65 32 24	6x	1+1
H2	48 47 45 36 33 65 63 24	6x	1+2
H3	48 47 44 36 33 65 62 24	6x	1+3
H4	48 47 43 36 33 65 61 24	6x	1+4
H5	48 47 42 36 33 65 30 24	6x	1+5
H6	48 46 43 36 33 64 61 24	6x	2+3

8-10. *Conjunct rationalized harmoniai. These harmoniai are formed in analogy to the conjunct Dorian of Schlesinger. The Hypodorian forms are based on the "bastard" harmonia. The lower tetrachords of Schlesinger's preferred form are 32 10 30 24, 48 47 46 36, 48 47 45 36, and 80 78 75 60.*

Mixolydian

DIATONIC

14 13 12 11 21 9 8 7

CHROMATIC

28 27 26 22 21 20 16 14

TRICHROMATIC 1

42 41 40 33 32 31 24 21

TRICHROMATIC 2

42 41 39 33 32 30 24 21

ENHARMONIC

56 55 54 44 43 42 32 28

PENTACHROMATIC

70 68 65 55 53 50 40 35

Lydian

DIATONIC

13 12 11 10 19 8 7 13

CHROMATIC

26 25 24 20 19 18 14 13

TRICHROMATIC 1

39 38 37 30 29 28 21 39

TRICHROMATIC 2

39 38 36 30 29 27 21 39

ENHARMONIC

52 51 50 40 39 38 28 26

PENTACHROMATIC

65 63 60 55 50 48 45 65

Phrygian

DIATONIC

24 22 20 18 17 14 13 6

CHROMATIC

24 23 22 18 17 16 13 12

TRICHROMATIC 1

36 35 34 27 26 25 39 18

TRICHROMATIC 2

36 35 33 54 26 24 39 18

ENHARMONIC

48 47 46 36 35 34 26 24

PENTACHROMATIC

60 58 55 45 40 38 65 30

Dorian

DIATONIC

11 10 9 8 15 13 6 11

CHROMATIC

22 21 20 16 15 14 12 11

TRICHROMATIC 1

33 32 31 24 23 22 18 33

TRICHROMATIC 2

33 32 30 24 23 21 18 33

ENHARMONIC

44 43 42 32 31 30 24 22

PENTACHROMATIC

55 53 50 40 35 33 30 55

Hypolydian

DIATONIC

20 18 16 15 13 12 11 10

CHROMATIC

20 19 18 15 14 13 11 10

TRICHROMATIC 1

60 58 56 45 43 41 33 30

TRICHROMATIC 2

60 58 54 45 43 39 33 30

40 39 38 30 29 28 22 20

PENTACHROMATIC

50 48 45 75 65 65 55 50

Hypophrygian

DIATONIC

18 16 15 13 25 11 10 9

CHROMATIC

18 17 16 13 25 12 10 9

TRICHROMATIC 1

54 52 50 39 38 37 30 27

TRICHROMATIC 2

54 52 48 39 38 36 30 27

ENHARMONIC

36 35 34 26 51 25 20 18

PENTACHROMATIC

90 86 80 65 63 60 50 45

Hypodorian

DIATONIC

16 15 13 12 23 10 9 8

CHROMATIC

32 30 28 24 23 22 18 16

TRICHROMATIC 1

48 46 44 36 35 34 27 24

TRICHROMATIC 2

48 46 42 36 35 33 27 24

ENHARMONIC

64 62 60 48 47 46 36 32

PENTACHROMATIC

80 76 70 60 58 55 45 40

8-11. *Synopsis of the rationalized tonoi. The tonoi are transpositions of the Dorian modal sequence so that the modal determinant of each harmonia falls on hypate meson. A local trite synemmenon has been defined in each of these harmoniai. In the Hypolydian, 15 alternates with 14. When mese falls on 14, trite synemmenon is 27 (27/22). The Hypodorian also has a "bastard" form which runs from proslambanomenos to mese in the Dorian tonos. The first tetrachord is 16 14 13 12.*

NAME	P	HH	HM	M	TS	P	ND
MIXOLYDIAN	44	40	28	22	21	20	14
LYDIAN	40	36	26	20	19	18	13
PHRYGIAN	36	32	24	18	17	16	12
DORIAN	32	28	22	16	15	14	11
HYPOLYDIAN	28	26	20	15/2	14	13	10
HYPOPHRYGIAN	26	24	18	13	25/2	12	9
HYPODORIAN	24	22	16	12	23/2	11	8

New conjunct forms

The new tritai synemmenon combine with the remaining tones to yield conjunct forms for each of the harmoniai. In order to preserve genera-specific melodic contours, a variation on the usual principle of construction was employed in the derivation of these scales. The procedure may be thought of as a type of inverse katapyknosis utilizing the note alternative to the local trite synemmenon in some cases. These conjunct harmoniai are listed in 8-10 in their diatonic, various chromatic, and enharmonic forms. The tuning of the principal structural notes of the rationalized tonoi is summarized in 8-11.

New modal determinants

As mentioned previously, one of the most noticeable inconsistencies in Schlesinger's system is the lack of a harmonia whose modal determinant is 15. Similarly in the new conjunct harmoniai, modal determinants of 17, 19, 21, 23, and 25 are implied by the local tritai synemmenon of the rationalized tonoi. Schlesinger herself stipulates the existence of harmoniai on 21 and 27 as later modifications of the Dorian and Lydian harmoniai. She claimed that these harmoniai were created by shifting their modal determinants one degree lower.

Additional harmoniai on modal determinants 29 and 31 may be added without exceeding the bounds of the Perfect Immutable System. To these may be added a harmonia on 33, which, though it exceeds the boundaries of the Dorian tonos, is included in the ranges of the tonoi of 8-12 and 8-13. The normal or disjunct forms of these new harmoniai are shown in 8-12 and the conjunct, which use their local tritai synemmenon, in 8-13. A summary of these new harmoniai is given in 8-14.

8-12 (next page). *New harmoniai. These harmoniai were created to fill in the gaps in Schlesinger's system, although some, such as tonoi-15, -21, and -27, are implied in her text. Three new genera are also provided; these are based on katapyknosis by 3 and 5 instead of 2 and 4. In principle, 14 may be substituted for 15 in these harmonia, save for tonos-15 where the Mixolydian harmonia would result. Similarly, 21 may replace 22 and 27, 26, except when doing so would change the modal determinant. In the diatonic genus when the first interval above the modal determinant is roughly a semitone, chromatic alternation with the next highest degree would be melodically acceptable.*

Tonos-15

DIATONIC

15 13 12 11 10 9 8 15

CHROMATIC

15 14 13 11 10 19 9 15

TRICHROMATIC 1

45 44 43 33 30 29 28 45

TRICHROMATIC 2

45 44 42 33 30 29 27 45

ENHARMONIC

30 29 28 22 20 39 19 15

PENTACHROMATIC

75 71 65 55 50 48 45 75

Tonos-17

DIATONIC

17 15 13 12 11 10 9 17

CHROMATIC

17 16 15 12 11 21 10 17

TRICHROMATIC 1

51 49 47 36 33 32 31 51

TRICHROMATIC 2

51 49 45 36 33 32 30 51

ENHARMONIC

34 33 32 24 22 43 21 17

PENTACHROMATIC

85 81 75 60 55 53 50 85

Tonos-19

DIATONIC

19 18 16 14 13 12 11 19

CHROMATIC

19 18 17 14 13 25 12 19

TRICHROMATIC 1

57 55 53 42 39 38 37 57

TRICHROMATIC 2

57 55 51 42 39 38 36 57

ENHARMONIC

38 37 36 28 26 51 25 19

PENTACHROMATIC

95 91 85 70 65 63 60 95

Tonos-21

DIATONIC

21 19 18 16 14 13 12 21

CHROMATIC

21 20 19 16 14 27 13 21

TRICHROMATIC 1

63 61 59 48 42 41 40 63

TRICHROMATIC 2

63 61 57 48 42 41 39 63

ENHARMONIC

42 41 40 32 28 55 27 21

PENTACHROMATIC

105 101 95 80 70 68 65 105

Tonos-23

DIATONIC

23 21 20 18 16 14 13 23

CHROMATIC

23 22 21 18 16 15 14 23

TRICHROMATIC 1

69 67 65 54 48 46 44 69

TRICHROMATIC 2

69 67 63 54 48 46 42 69

ENHARMONIC

46 45 44 36 32 31 30 23

PENTACHROMATIC

115 111 105 90 80 76 70 115

Tonos-25

DIATONIC

25 22 20 18 16 14 13 25

CHROMATIC

50 47 22 36 32 30 28 25

TRICHROMATIC 1

75 72 69 54 48 46 44 75

TRICHROMATIC 2

75 72 66 54 48 46 42 75

ENHARMONIC

50 97 47 36 32 31 30 25

PENTACHROMATIC

125 119 110 90 80 76 70 125

Tonos-27

DIATONIC

27 24 21 20 18 16 14 27

CHROMATIC

54 51 48 40 36 34 32 27

TRICHROMATIC 1

81 78 75 60 54 52 50 81

TRICHROMATIC 2

81 78 72 60 54 52 48 81

ENHARMONIC

101 102 40 36 35 34 54

PENTACHROMATIC

135 129 120 100 90 86 80 135

Tonos-29

DIATONIC

29 26 24 22 20 18 16 29

CHROMATIC

29 28 27 22 20 19 18 29

TRICHROMATIC 1

87 85 83 66 60 58 56 87

TRICHROMATIC 2

87 85 81 66 60 58 54 87

ENHARMONIC

58 57 56 44 40 39 38 29

PENTACHROMATIC

145 141 135 110 100 96 90 145

Tonos-31

DIATONIC

31 28 26 23 22 20 18 31

CHROMATIC

31 29 27 23 22 21 20 31

TRICHROMATIC 1

93 89 85 69 66 64 62 93

TRICHROMATIC 2

93 89 81 69 66 64 60 93

ENHARMONIC

31 30 29 23 22 43 21 31

PENTACHROMATIC

155 147 135 115 110 106 100 155

Tonos-33

DIATONIC

33 30 27 24 22 20 18 33

CHROMATIC

33 31 29 24 22 21 20 33

TRICHROMATIC 1

99 96 93 72 66 64 62 99

TRICHROMATIC 2

99 96 90 72 66 64 60 99

ENHARMONIC

33 32 31 24 22 43 21 33

PENTACHROMATIC

165 159 150 120 110 106 100 165

Tonos-21: Schlesinger claimed that the Dorian 22 was lowered in the PIS to 21 and that of the Lydian from 27 to 26; tonos-21 is thus the Dorian of the PIS. Tonos-25: It has proven difficult to obtain harmoniai whose melodic forms are characteristic of the genera. This tonos demands chromatic alternatives (17 for 16, 48 for 47, 23 for 22, 97 for 98, etc.). Tonos-27: This was conjectured by Schlesinger to be the Syntonolydian. Note 21 may alternate with 22. It may be described as the Lydian of the PIS. Alternative forms are 27 24 22 20 18 16 14 27, 27 26 25 20 18 17 16 27, and 54 53 52 40 36 35 34 27. Tonos-29: In the diatonic, 26 may alternate with 27. Tonos-31: These harmoniai admit several variants where 24 and 23, 29 and 30, 28 and 27 are alternatives. In tonos-33, the diatonic has a variant 33 29 27 24, the chromatic 33 63 30 24, the first trichromatic 99 95 91 72, the second trichromatic 99 95 87 72, and the pentachromatic 165 157 145 120 110.

Tonos-15

DIATONIC

15 13 12 11 21 18 16 15

CHROMATIC

15 14 13 11 21 20 16 15

TRICHROMATIC 1

45 44 43 33 32 31 24 45

TRICHROMATIC 2

45 44 42 33 32 30 24 45

ENHARMONIC

30 29 28 22 43 21 16 15

PENTACHROMATIC

75 71 65 55 53 50 40 75

Tonos-17

DIATONIC

17 15 13 12 23 10 9 17

CHROMATIC

17 16 15 12 23 11 9 17

TRICHROMATIC 1

51 49 47 36 35 34 27 51

TRICHROMATIC 2

51 49 45 36 35 33 27 51

ENHARMONIC

34 33 32 24 47 23 18 17

PENTACHROMATIC

85 81 75 60 58 55 90 85

Tonos-19

DIATONIC

19 18 16 14 27 12 11 19

CHROMATIC

19 18 17 14 27 13 11 19

TRICHROMATIC 1

57 55 53 42 41 40 33 57

TRICHROMATIC 2

57 55 51 42 41 39 33 57

ENHARMONIC

38 37 36 28 55 54 22 19

PENTACHROMATIC

95 91 85 70 68 65 55 95

Tonos-21

DIATONIC

21 19 18 16 15 13 12 21

CHROMATIC

21 20 19 16 15 14 12 21

TRICHROMATIC 1

63 61 59 48 46 44 36 63

TRICHROMATIC 2

63 61 57 48 46 42 36 63

ENHARMONIC

42 41 40 32 31 30 24 21

PENTACHROMATIC

105 101 95 80 76 70 60 105

Tonos-23

DIATONIC

23 21 20 18 17 14 13 23

CHROMATIC

23 22 21 18 17 16 13 23

TRICHROMATIC 1

69 67 65 54 52 50 39 69

TRICHROMATIC 2

69 67 63 54 52 48 39 69

ENHARMONIC

46 45 44 36 35 34 26 23

PENTACHROMATIC

115 111 105 90 86 80 65 115

Tonos-25

DIATONIC

25 22 20 18 17 14 13 25

CHROMATIC

50 47 44 36 34 32 26 25

TRICHROMATIC 1

75 72 69 54 52 50 39 75

TRICHROMATIC 2

75 72 66 54 52 48 39 75

ENHARMONIC

50 97 47 36 35 34 36 25

PENTACHROMATIC

125 119 110 90 86 80 65 125

Tonos-27

DIATONIC

27 24 21 20 19 16 14 27

CHROMATIC

54 51 48 40 38 36 28 27

TRICHROMATIC 1

81 78 75 60 58 56 42 81

TRICHROMATIC 2

81 78 72 60 58 54 42 81

ENHARMONIC

54 105 51 40 39 38 28 27

PENTACHROMATIC

135 129 120100 96 90 70 135

Tonos-29

DIATONIC

29 26 24 22 21 18 16 29

CHROMATIC

29 28 27 22 21 20 16 29

TRICHROMATIC 1

87 85 83 66 64 62 48 87

TRICHROMATIC 2

87 85 81 66 64 60 48 87

ENHARMONIC

58 57 56 44 43 42 32 29

PENTACHROMATIC

145 141 135 110 106 100 80 145

Tonos-31

DIATONIC

31 28 26 23 22 20 18 31

CHROMATIC

31 29 27 23 22 21 18 31

TRICHROMATIC 1

93 89 85 69 67 65 54 93

TRICHROMATIC 2

93 89 81 69 67 63 54 93

ENHARMONIC

31 30 29 23 45 44 36 31

PENTACHROMATIC

155 147 135 115 111 105 90 155

Tonos-33

DIATONIC

33 30 27 24 23 20 18 33

CHROMATIC

33 31 29 24 23 22 18 33

TRICHROMATIC 1

99 96 93 72 70 68 54 99

TRICHROMATIC 2

99 96 90 72 70 66 54 99

ENHARMONIC

33 32 31 24 47 46 18 33

PENTACHROMATIC

165 159 150 120 116 110 90 165

8-13. *New conjunct harmoniai. In this context, conjunct means employing the local tonos-specific trite synemmenon.*

8-14. *Synopsis of the new tonoi. The tonoi are transpositions of the Dorian modal sequence so that the determinant of each harmonia falls on hypate meson. A local trite synemmenon for each of the harmoniai has been defined. Certain odd or prime number modal determinants have been expressed as fractions, i.e. 21/2, to indicate the higher octave since the modal determinants represent aliquot parts of vibrating air columns or strings. Modal determinants 14 (28) and 15 (30) are alternates. Tonos-31: in the conjunct form, mese is 23, trite synemmenon is 22.*

	P	HH	HM	M	TS	P	ND
TONOS-15	22	20	15	11	21/2	10	15/2
TONOS-17	24	22	17	12	23/2	11	17/2
TONOS-19	28	26	19	14	27/2	13	19/2
TONOS-21	32	28	21	16	15	14	21/2
TONOS-23	36	32	23	18	17	16	23/2
TONOS-25	36	32	25	18	17	16	25/2
TONOS-27	40	36	27	20	19	18	27/2
TONOS-29	44	40	29	22	21	20	29/2
TONOS-31	48	44	31	24	22	22	31/2
TONOS-33	48	44	33	24	23	22	33/2

Harmonizing the new harmoniai

The new harmoniai may be harmonized by methods analogous to those Elsie Hamilton employed with Schlesinger's diatonic harmoniai. The tetrachordal framework chords of both the disjunct and conjunct forms of the new harmoniai are shown in 8-15.

The framework chords from the new conjunct forms are particularly interesting harmonically as they provide a means of incorporating the new harmoniai with the older system. Because many of the modal determinants of the new harmonia are prime numbers, their tetrachordal framework chords do not share many notes with the ones from the older scales. Certain chords, however, from the new conjunct harmoniai do share notes with the framework chords of the older forms and thus allow one to modulate by common tone progressions. These chords may also be used in progressions similar to those in 8-6c and 8-7.

Moreover, these chords may be used to harmonize the mesopykna of the chromatic harmoniai and the oxypykna of the enharmonic which seemingly lay outside of Hamilton's harmonic concerns.

Harmoniai with more than seven tones

Although it is quite feasible to define harmoniai with modal determinants between 33 and 44 (the limit of the Mixolydian tonos), it becomes increasingly difficult to decide the canonical forms such harmoniai might take because of the rapidly increasing number of chromatic or alternative tones available in the octave.

Rather than omit the extra tones in these and the harmoniai with smaller modal determinants, one may define harmoniai with more than seven tones and utilize the resulting melodic and harmonic resources.

8-15. *Harmonization of the new harmoniai. Tetrachordal framework chords.*

	DISJUNCT	CONJUNCT
HARMONIA-15	15:11:10:15/2	15:11:8:15/2
HARMONIA-17	17:12:11:17/2	17:12:9:17/2
HARMONIA-19	19:14:13:19/2	19:14:11:19/2
HARMONIA-21	21:16:14:21/2	21:16:12:21/2
HARMONIA-23	23:18:16:23/2	23:18:13:23/2
HARMONIA-25	25:18:16:25/2	25:18:13:25/2
HARMONIA-27	27:20:18:27/2	27:20:14:27/2
HARMONIA-29	29:22:20:29/2	29:22:16:29/2
HARMONIA-31	31:24:22:31/2, 31:23:22:31/2	31:23:18:31/2, 31:24:18:31/2
HARMONIA-33	33:24:22:33/2	33:24:18:33/2

8-16. *Harmonic forms of the Phrygian harmonia. For each of the diatonic harmoniai, the harmonic forms are obtained by taking the 2/1 complement of each ratio or interval.*

FIRST VERSION OF THE INVERTED PHRYGIAN

DIATONIC

12 13 14 16 18 20 22 24

CHROMATIC

12 14 15 16 18 22 23 24

ENHARMONIC

24 30 31 32 36 46 47 48

SECOND VERSION OF THE INVERTED PHRYGIAN

CHROMATIC

24 25 26 32 36 38 40 48

ENHARMONIC

48 49 50 64 72 74 76 96

8-17. *Harmonic forms of the conjunct Phrygian harmonia. For each of the conjunct diatonic harmoniai, the harmonic form is obtained by taking the 2/1 complement of each ratio or interval.*

FIRST VERSION OF THE INVERTED CONJUNCT PHRYGIAN HARMONIAI

DIATONIC

12 13 14 17 18 20 22 24

CHROMATIC

12 13 16 17 18 22 23 24

ENHARMONIC

24 26 34 35 36 46 47 48

SECOND VERSION OF THE INVERTED CONJUNCT PHRYGIAN HARMONIAI

CHROMATIC

24 26 27 28 36 38 40 48

ENHARMONIC

48 52 53 54 72 74 76 96

Another source of new harmoniai has been suggested by Wilson. One might insert pykna above notes other than the first and fourth degrees of the basic diatonic modal sequence. Interesting variations may also be discovered by inserting more than two pykna, or any number at any location. The final result of this procedure is to generate "close-packed" scales with many more than seven notes.

Harmonic forms of the harmoniai

Schlesinger's original harmoniai and all of the new scales generated in analogy with hers are 1- or 2-octave sections of the subharmonic series. These musical structures may be converted to sections of the harmonic series by replacing each of their tones with their 2/1 complements or octave inversions.

The resulting harmonic forms may be used in exactly the same way as the originals, save that the modalities of the chords (major or minor) and the melodic contours of the scales are reversed, i.e., the intervals become smaller rather than larger as one ascends from the lowest tone.

In general, chords from the harmonic series are more consonant than those from the subharmonic. However, the tones of the harmonic scales are more likely to be heard as arpeggiated chords than are the scalar tones of the subharmonic forms.

There is only one form of each of the inverted diatonic harmoniai, but the chromatic, enharmonic and other katapyknotic forms (8-9) have two versions. The first forms are the octave complements of the corresponding subharmonic originals and these forms have their pykna at the upper end of each tetrachord. The second versions are produced by dividing the initial intervals of the two tetrachords of the inverted diatonic forms as in the generation of the chromatic and other katapyknotic forms of 8-9. An example which illustrates these operations is shown in 8-16. The Phrygian harmonia, of modal determinant 12, is inverted and then divided to yield the diatonic, chromatic and enharmonic forms. Both versions of the chromatic and enharmonic harmoniai are listed, and the other katapyknotic forms may be obtained by analogy.

Conversely, the second of the new harmonic forms may be inverted to derive new subharmonic harmoniai whose divided pykna lie at the top of their tetrachords. These too are listed in 8-16.

Conjunct harmoniai may also be inverted to generate harmonic

8-18. *Wilson's diaphonic cycles. These diaphonic cycles (diacycles) may be constructed on sets of strings tuned alternately a 3/2 and 4/3 apart since the largest divided interval is the 3/2. The order of the segments, nodes, and conjunctions may be permuted according to the following scheme: a/b · c/d = a/d · c/b = 2/1 and c/d · a/b = c/b · a/d = 2/1. Alternative conjunctions are indicated by primed nodes, i.e. c′, d′. Some diacycles such as number 21 have two independent sets of nodes and conjunctions. The second is symbolized by e f g h.*

1. 9 8 7 6
a c b, d
(3/2 · 4/3)

2. 12 11 10 9 8
a, c d b
(3/2 · 4/3)

3. 18 17 16 15 14 13 12
a c b, d
(3/2 · 4/3)

4. 21 20 19 18 17 16 15 14
a c d b
(3/2 · 4/3; 10/7 · 7/5)

5. 24 23 22 21 20 19 18 17 16
a, c b d
(3/2 · 4/3)

6. 27 26 25 24 23 22 21 20 19 18
a c b, d
(3/2 · 4/3)

7. 30 ……… 28 ……… 21 20
a c d b
(3/2 · 4/3; 10/7 · 7/5)

8. 33 32 ……… 24 ……… 22
a c d b
(3/2 · 4/3; 16/11 · 11/8)

9. 36 ……… 32 ……… 27 ……… 24
a, c c′ d b
(3/2 · 4/3)

10. 39 ……… 36 ……… 27 26
a c d b
(3/2 · 4/3; 13/9 · 18/13)

11. 42 ……… 40 ……… 30 ……… 28
a c d b
(3/2 · 4/3; 10/7 · 7/5)

12. 45 44 ……… 40 ……… 33 ……… 30
a c′ c d′ b, d
(3/2 · 4/3; 22/15 · 15/11)

13. 48 ……… 44 ……… 36 ……… 33 32
a, c′ c d′ d b
(3/2 · 4/3; 16/11 · 11/8)

14. 51 ……… 48 ……… 36 ……… 34
a c d b
(3/2 · 4/3; 17/12 · 24/17)

15. 54 ……… 52 ……… 48 ……… 39 ……… 36
a c′ c d′ b, d
(3/2 · 4/3; 13/9 · 18/13)

16. 57 56 ……… 52 ……… 42 ……… 39 38
a c′ c d′ d b
(3/2 · 4/3; 19/14 · 28/19; 19/13 · 26/19)

17. 60 ……… 56 ……… 42 ……… 40
a c d b
(3/2 · 4/3; 10/7 · 7/5)

18. 63 ……… 60 ……… 56 ……… 45 ……… 42
a c′ c d′ b, d
(3/2 · 4/3; 10/7 · 7/5)

19. 66 ……… 64 ……… 60 ……… 48 ……… 45 44
a c′ c d′ d b
(3/2 · 4/3; 22/15 · 15/11; 16/11 · 11/8)

20. 69 68 ……… 64 ……… 51 ……… 48 ……… 46
a c′ c d′ d b
(3/2 · 4/3; 23/16 · 32/23; 23/17 · 34/23)

21. 72 ……… 70 ……… 68 ……… 64 ……… 51 50 49 48
a e, g c′ c d′ h f b, d
(3/2 · 4/3; 10/7 · 7/5; 24/17 · 17/12)

22. 75 ……… 68 ……… 51 50
a c d b
(3/2 · 4/3; 25/17 · 34/25)

23. 78 ……… 76 ……… 57 ……… 52
a c d b
(3/2 · 4/3; 26/19 · 19/13)

24. 81 80 ……… 77 ……… 60 ……… 56 55 54
a c, e g d h f b

8-19. *Diacycles on 20/13. These diacycles can be constructed on strings 13/10 and 20/13 apart.*

40 39 … 36 ………… 30 … 27 26
a *c,e* *g* *d* *f* *b,b*
(20/13 · 13/10; 3/2 · 4/3; 13/9 · 18/13)

60 … 56 … 52 ………… 42 .. 40 39
a,e *g* *c* *f,b* *d* *b*
(20/13 · 13/10; 3/2 · 4/3; 10/7 · 7/5)

80 .. 78 … 76 ………… 60 … 57 …… 52
a *c,e* *g* *d* *f* *b,b*
(20/13 · 13/10; 3/2 · 4/3; 26/19 · 19/13)

100 99 … 96 …. 91 ………… 72 … 70 ….. 66 65
a *e* *g* *c* *h* *d* *f* *b*
(20/13 · 13/10; 10/7 · 7/5; 3/2 · 4/3; 16/11 · 11/8)

8-20. *Triaphonic and tetraphonic cycles on 4/3 and 5/4. (1) may be constructed on three strings tuned to 1/1, 4/3, and 3/2. (2) requires strings tuned to 1/1, 4/3, and 3/2. (3) may be realized on four strings tuned to 1/1, 6/5, 147/100 and 42/25.*

20 19 18 17 16 15
a, *c* *e* *d* *b,* *f*
(4/3 · 5/4 · 6/5)

28 27 ………… 24 ………… 21
a, c *e* *d* *b, f*
(4/3 · 7/6 · 9/7)

50 49 48 ………… 42 …… 40
a *e* *c, g* *f, b* *b, d*
(5/4 · 6/5 · 7/6 · 8/7)

forms as shown in 8-17. In this case, the disjunctive tone is at the bottom with the two tetrachords linked by conjunction above.

These operations may be applied to all of the harmoniai described above. Similarly, the other musical structures presented in the remainder of this chapter may also be inverted.

Other directions: Wilson's diaphonic cycles

Ervin Wilson has developed a set of scales, the *diaphonic cycles,* which combine the repeated modular structure of tetrachordal scales with the linear division of Schlesinger's harmoniai (Wilson, personal communication).

The diaphonic cycles, or less formally *diacycles,* may be understood most easily by examining the construction of the two simplest members in 8-18.

In diacycle 1, the interval 3/2, which is bounded by the nodes *a* and *b,* is divided linearly to generate the subharmonic sequence 9 8 7 6 or 1/1 9/8 9/7 3/2. Subtended by this 3/2 is the linearly divided 4/3 bounded by the nodes *c* and *d.* This segment forms the sequence 8 7 6 or 1/1 8/7 4/3. Five-tone scales may be produced by joining these two melodic segments with a common tone to yield 1/1 9/8 9/7 3/2 12/7 2/1 (*a* – *b* on 1/1, then *c*–*d* on 3/2) and 1/1 8/7 4/3 3/2 12/7 2/1 (*c*–*d* on 1/1, then *a*–*b* on 4/3):

9 8 7 (6) and 8 7 (6)
(8) 7 6 (9) 8 7 6

The tones in parentheses are common to the two segments.

Diaphonic cycle 2 generates two heptatonic scales which are modes of Ptolemy's equable diatonic genus: 1/1 12/11 6/5 4/3 16/11 8/5 16/9 2/1 and 1/1 12/11 6/5 4/3 3/2 18/11 9/5 2/1. The two forms are respectively termed the conjunctive and disjunctive or tetrachordal form.

As the linear division becomes finer, scales with increasing numbers of tones are generated. At number 4, a new phenomenon emerges: the existence of another set of segments whose conjunction produces complete scales. The nodes *a,d* and *c,b* define a pair of diaphonic cycles whose segments are 10/7 and 7/5.

These diaphonic cycles can be implemented on instruments such as guitars by tuning the intervals between the strings to a succession of 3/2's and 4/3's. The fingerboards must be refretted so that the frets occur at equal aliquot parts of the string length. Wilson constructed several such guitars in the early 1960s.

8-21. *Divisions of the fifth. (1) is described as an "aulos-scale (Phrygian, reconstructed by KS)" in Schlesinger 1933. (2) is another "aulos-scale (Hypodorian)," identified with another unnamed scale of Aristoxenos (Meibomius 1652, 72). (3) is an "aulos-scale (Mixolydian)," identified with another unnamed scale of Aristoxenos. (4) is identified with yet another scale of Aristoxenos. (5) spans an augmented fifth and appears also in her interpretation of the spondeion. (6) is the "singular major" of Safiyu-d-Din (D'Erlanger 1938, 281). The Islamic genera are from Rouanet 1922. (8), Isfahan, spans only the 4/3. (9) is labeled "Zirafkend Bouzourk." Rouanet's last genus is identical to Safiyu-d-Din's scale of the same name.*

SCHLESINGER'S DIVISIONS

1. 24/23 · 23/22 · 11/9 · 9/8
2. 16/15 · 15/14 · 7/6 · 9/8
3. 28/27 · 9/8 · 8/7 · 9/8
4. 21/20 · 10/9 · 9/8 · 8/7
5. 11/10 · 10/9 · 9/8 · 8/7

ISLAMIC GENERA

6. 14/13 · 8/7 · 13/12 · 14/13 · 117/112
7. 13/12 · 14/13 · 13/12 · 287/272
8. 13/12 · 14/13 · 15/14 · 16/15
9. 14/13 · 13/12 · 36/35 · 9/8 · 10/9

Wilson has also developed a set of simpler scales on the same principles under the general name of "Helix Song." They consist of notes selected from the harmonic series on the tones 1/1 and 4/3. These have been used as the basis of a composition by David Rosenthal (Rosenthal 1979).

Triacycles and tetracycles

For the sake of completeness, some new diacycles have been constructed on the interval pair 20/13 and 13/10. These are listed in 8-19. As 20/13 is slightly larger than 3/2, some new diacycles on 3/2 are generated incidentally too.

Larger intervals and their octave complements might be used, but the increased inequality in the sizes of the two segments would probably be melodically unsatisfactory. This asymmetry may be hidden by defining three or four segments instead of merely two. A few experimental three- and four-part structures, which may be called *triacycles* and *tetracycles,* are shown in 8-20.

Linear division of the fifth

As a final note, it must be mentioned that both Schlesinger (1933) and the Islamic theorists also recognized scales derived by linear division of the fifth instead of the fourth or octave (8-21). Not surprisingly, Schlesinger's are presented as support for the authenticity of her harmoniai.

It is likely that the Islamic forms had origins that are independent of the Greek theoretical system. The genus from Safiyu-d-Din (D'Erlanger 1938) may be rationalized as being derived from the permuted tetrachord, 14/13 · 8/7 · 13/12, by dividing the disjunctive tone, 9/8, of the octave scale into two unequal parts, 14/13 and 117/112. Characteristically, all 24 permutations of the intervals were tabulated.

Rouanet's scales deviate even more from Greek models, though the tetrachordal relationship may still be seen (Rouanet 1922).

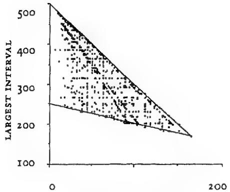
9 The Catalog of tetrachords

THIS CATALOG ATTEMPTS a complete and definitive compilation of all the tetrachords described in the literature and those that can be generated by the straightforward application of the arithmetic and geometric concepts described in the previous chapters. While the first of these goals can be achieved in principle, the second illustrates Aristoxenos's tenet that the divisions of the tetrachord are potentially infinite in number. It seems unlikely, however, that any great number of musically useful or theoretically interesting tetrachords has been omitted. Figures 9-1 through 9-6 show that the two-dimensional tetrachordal space is nearly filled by the tetrachords in the Catalog. The saturation of perceptual space is especially likely when one considers the finite resolving power of the ear, the limits on the accuracy and stability of analog and acoustic instruments, the quantizing errors of digital electronics, and our readiness to accept sufficiently close approximations to ideal tunings.

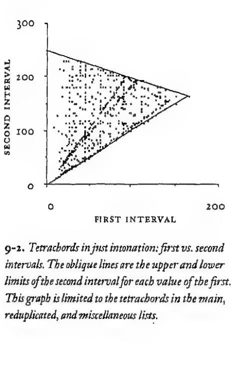
Nevertheless, processes such as searches through large microchromatic scales (chapter 7) and propriety calculations (chapter 5) will occasionally turn up new genera, so perhaps one should not be too complacent. The great majority of these new tetrachords, however, will resemble those already in the Catalog or be interchangeable with them for most melodic and harmonic purposes.

Organization of the Catalog

The tetrachords in the Main Catalog are listed by the size of their largest interval, which, in lieu of an historically validated term, has been called the



9-1. *Tetrachords in just intonation: smallest vs. largest intervals. Units in cents. The oblique lines are the upper and lower limits of the largest interval for each value of the smallest. This graph is limited to the tetrachords in the main, reduplicated, and miscellaneous lists.*



9-2. *Tetrachords in just intonation: first vs. second intervals. The oblique lines are the upper and lower limits of the second interval for each value of the first. This graph is limited to the tetrachords in the main, reduplicated, and miscellaneous lists.*

characteristic interval (CI). The term *apyknon* would have been used except that it has been traditionally employed to denote the sum of the two lower intervals of the diatonic genera. In diatonic tetrachords, this sum is greater than one half of the fourth.

Those tetrachords with CIs larger than 425 cents are classed as hyperenharmonic (after Wilson) and listed first. Next come the enharmonic with their *incomposite* CIs approximating major thirds. Chromatic and diatonic genera follow, the latter beginning when the CI falls below 250 cents.

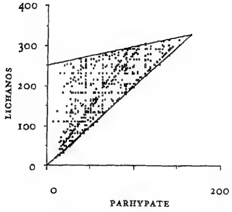
For each CI, the genera derived from the 1:1, 1:2, and 2:1 divisions of the pyknon or apyknon are listed first and followed by the other species of tetrachord with this CI. References to the earliest literature source and a brief discussion of the genus are given below each group.

In addition to the genera from the literature, the majority of the Main Catalog comprises tetrachords generated by the processes outlined in chapters 4 and 5. Both the 1:2 and 2:1 divisions are provided because both must be examined to select "strong," mostly superparticular forms in the Ptolemaic manner (chapter 2). If strict superparticularity is less important than convenience on the monochord or linear order, the 1:2 division is preferable, but recourse to the 2:1 may be necessary to discover the simplest form. For example, the threefold division of the 16/15 pyknon yields the notes 48 47 46 45. Ptolemy chose to recombine the first two intervals and reorder the third to obtain his enharmonic, 46/45 · 24/23 · 5/4.

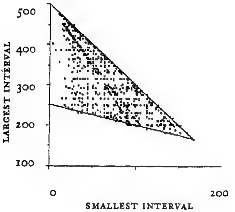
In general, only the simplest or mostly superparticular divisions are tabulated in this section; occasionally a theoretically interesting tetrachord without any near relatives will be found in the Miscellaneous list. Such isolated tetrachords are relatively uncommon. There are cases, however, in which all of the other divisions of a tetrachord's pyknon or apyknon have very complex ratios, and so closely resemble other tetrachords already tabulated that it did not seem fruitful to list them in a group under the CI in the Main Catalog.

"Miscellaneous" is a very elastic category. It consists of a collection of genera of diverse origin that I did not think interesting enough to list in the Main Catalog.

The order of intervals within each tetrachord is the canonical small, medium, and large in the case of the historical genera and their analogs. The new theoretical genera are generally listed in the order resulting from



9-3. *Tetrachords in just intonation: parhypatai vs. lichanoi. The oblique lines are the upper and lower limits of lichanos for each value of parhypate. This graph is limited to the tetrachords in the main, reduplicated, and miscellaneous lists.*



9-4. *Just and tempered tetrachords: smallest vs. largest intervals. The oblique lines are the upper and lower limits of the largest interval for each value of the smallest. This graph contains all the tetrachords in the Catalog.*

their generating process. It should be remembered, however, that all six permutations of the non-reduplicated genera and all three of the reduplicated are equally valid for musical experimentation.

With the exception of the Pythagorean 256/243 · 9/8 · 9/8 and Al-Farabi's 10/9 · 10/9 · 27/25, the genera with reduplicated intervals are given in the list of Reduplicated tetrachords.

Those tetrachords defined in either in "parts" of the tempered fourth or which consist solely of tempered intervals are to be found in the Tempered list. Needless to say, these tetrachords are a diverse lot, covering Aristoxenos's divisions, Greek Orthodox liturgical genera (in two systems — one of 28 parts to the fourth, the other of 30), and those derived from theoretical considerations. As some of the latter contain rational intervals as well, a separate list of Semi-tempered tetrachords is included.

No attempt has been made to catalog the very numerous tetrachords and tetrachord-like structures found in the non-zero modulo 12 equal temperaments of 4-17.

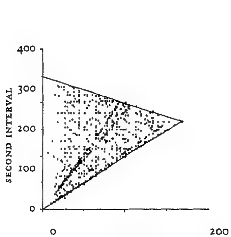
An index of sources for those tetrachords of historical provenance is provided.

Uniformity of sampling

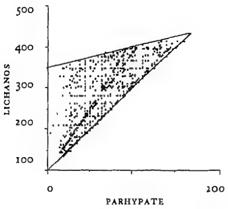
In order to show the uniformity with which the set of all possible tetrachords in just intonation has been sampled in the Catalogs of this chapter, the genera from the Main, Reduplicated, and Miscellaneous lists have been plotted in 9-1, 9-2 and 9-3. In 9-1, the smallest intervals are plotted against the largest intervals or CIs. As one may see, the area delineated by the two oblique lines is more or less uniformly filled. However, diagonal zones corresponding to genera with roughly equal and 1:2 divisions are evident. The tables are deliberately deficient in genera with commatic and sub-commatic intervals, as these are of little use melodically. The few examples in the tables are taken mostly from Hofmann's list of superparticular divisions (Vogel 1975) or generated by theoretical operations such as the means of chapter 4.

9-2 is a plot of the first versus the second intervals of the same tetrachords. Although the graph has a different shape, the same conclusions may be drawn.

9-3 is a third representation of the same data. In this case, cumulative rather than sequential intervals have been plotted. This mode reflects the Greek classification of tetrachords into primary genera (enharmonic,



9-5. Just and tempered tetrachords: first vs. second intervals. The oblique lines are the upper and lower limits of the second interval for each value of the first. This graph contains all the tetrachords in the Catalog.



9-6. Just and tempered tetrachords: parhypatai vs. lichanoi. The oblique lines are the upper and lower limits of lichanos for each value of the parhypate. This graph contains all the tetrachords in the Catalog.

chromatic and diatonic) and shades or nuances *(chroai)* of these genera. The primary distinction is based on the size of the uppermost interval, usually the CI except in Archytas's and Ptolemy's diatonics (28/27 · 8/7 · 9/8 and 16/15 · 9/8 · 10/9). The exact nuance or shade is then defined by the size of the first interval. The position of parhypate is equivalent to the size of the first interval and the position of lichanos is an inverse measure of the CI. This graph also reveals the relative uniformity of coverage and the excess of genera with 1:1 and 1:2 divisions.

The tetrachords in the Tempered and Semi-tempered lists were added to the set graphed in 9-1–3, and the entire collection replotted in 9-4–6. The largest empty spaces in the plots are thus filled. In a few cases, the gaps could be filled only by creating new genera specifically for this task. These have been marked in the Tempered tetrachord list.

The Main Catalog

HYPERENHARMONIC TETRACHORDS

H1. CHARACTERISTIC INTERVAL 13/10 454 CENTS

1	80/79 · 79/78 · 13/10	22 + 22 + 454	
2	60/49 · 118/117 · 13/10	29 + 15 + 454	
3	120/119 · 119/117 · 13/10	14 + 29 + 454	
4	100/99 · 66/65 · 13/10	17 + 26 + 454	WILSON

The 13/10 would appear to be the upper limit for a genus-defining CI simply because the pyknotic intervals become too small to be melodically useful, however perceptible they might remain. In general, tetrachords with intervals less than 20 cents or with overly complex ratios will be relegated to the Miscellaneous listing at the end of the Catalog proper, unless there is some compelling reason, such as historical or literary reference, illustration of theory, or the like, to include them. The pyknon of this hyperenharmonic genus is the 40/39 (44 cents), which is very close to the Pythagorean double comma of $3^{24}/2^{38}$. Number 4 is from the unpublished notes of Ervin Wilson. See also Miscellaneous.

H2. CHARACTERISTIC INTERVAL 35/27 449 CENTS

5	72/71 · 71/70 · 35/27	24 + 25 + 449	
6	108/107 · 107/105 · 35/27	16 + 33 + 449	
7	54/53 · 106/105 · 35/27	32 + 16 + 449	
8	64/63 · 81/80 · 35/27	27 + 22 + 449	

This genus divides the 36/35 (49 cents), an interval found in Archytas's enharmonic and Avicenna's chromatic. Number 8 is found in Vogel's tuning for the Perfect Immutable System (Vogel 1963, 1967) and Erickson's (1965) analysis of Archytas's system (see chapter 6).

H3. CHARACTERISTIC INTERVAL 22/17 446 CENTS

9	68/67 · 67/66 · 22/17	26 + 26 + 446	
10	51/50 · 100/99 · 22/17	35 + 17 + 446	
11	102/101 · 101/99 · 22/17	17 + 35 + 446	
12	85/84 · 56/55 · 22/17	20 + 31 + 446	WILSON

The pyknon of this hyperenharmonic genus is 34/33 (52 cents), a quartertone. The intervening genera with pykna between 39/38 and 35/34 have not so far yielded melodically interesting, harmonically useful, nor mathematically elegant divisions, but see Miscellaneous for examples. This genus is replete with intervals of 17.

H4. CHARACTERISTIC INTERVAL 128/99 445 CENTS

13	66/65 · 65/64 · 128/99	26 + 27 + 445
14	99/98 · 49/48 · 128/99	18 + 36 + 445
15	99/97 · 97/96 · 128/99	35 + 18 + 445

The pyknon of this genus is 33/32 (53 cents), the octave-reduced thirty-third harmonic and an approximate quarter-tone.

H5. CHARACTERISTIC INTERVAL 31/24 443 CENTS

16	64/63 · 63/62 · 31/24	27 + 28 + 443
17	96/95 · 95/93 · 31/24	18 + 37 + 443
18	48/47 · 94/93 · 31/24	36 + 19 + 443

This hyperenharmonic genus divides the 32/31 (55 cents), an interval used in Didymos's enharmonic.

H6. CHARACTERISTIC INTERVAL 40/31 441 CENTS

19	62/61 · 61/60 · 40/31	28 + 29 + 441
20	93/92 · 46/45 · 40/31	19 + 38 + 441
21	93/91 · 91/90 · 40/31	38 + 19 + 441

The pyknon of this genus is 31/30 (57 cents), an interval which occurs in Didymos's enharmonic.

H7. CHARACTERISTIC INTERVAL 58/45 439 CENTS

22	60/59 · 59/58 · 58/45	29 + 30 + 439
23	90/89 · 89/87 · 58/45	19 + 39 + 439
24	45/44 · 88/87 · 58/45	39 + 20 + 439

25	120/119 · 119/116 · 58/45	14 + 44 + 439	

The pyknon of this hyperenharmonic genus is 30/29 (59 cents).

H8. Characteristic Interval 9/7 435 cents

26	56/55 · 55/54 · 9/7	31 + 32 + 435	WILSON
27	42/41 · 82/81 · 9/7	42 + 21 + 435	
28	84/83 · 83/81 · 9/7	21 + 42 + 435	
29	64/63 · 49/48 · 9/7	27 + 36 + 435	
30	70/69 · 46/45 · 9/7	25 + 38 + 435	
31	40/39 · 91/90 · 9/7	44 + 19 + 435	
32	112/111 · 37/36 · 9/7	16 + 47 + 435	
33	81/80 · 2240/2187 · 9/7	22 + 41 + 435	
34	9/7 · 119/117 · 52/51	435 + 29 + 34	

The pyknon of this prototypical hyperenharmonic genus (Wilson, unpublished) is Archytas's diesis, 28/27 (63 cents). Melodically, this genus bears the same relation to Aristoxenos's soft chromatic as Aristoxenos's enharmonic does to his syntonic (intense) chromatic. Number 26 is Wilson's original "hyperenharmonic" tetrachord. Divisions 29 and 31 are interesting in that their first intervals make, respectively, an 8/7 and a 15/13 with the subtonics hyperhypate (diatonic lichanos meson) and mese, and proslambanomenos and diatonic paranete diezeugmenon as well. Tetrachord number 32 is a good approximation to a hypothetical 1 + 3 + 26 parts, 17 + 50 + 433 cents—see also number 25 above. Number 33 occurs in Vogel's (1963, 1967) PIS tuning. Number 34 is a summation tetrachord from chapter 4.

H9. Characteristic Interval 104/81 433 cents

35	54/53 · 53/52 · 104/81	32 + 33 + 433
36	81/79 · 79/78 · 104/81	43 + 22 + 433
37	81/80 · 40/39 · 104/81	22 + 44 + 433

The pyknon of this genus is 27/26 (65 cents). This division is melodically similar to the 9/7 genus, though not harmonically. Number 37, when rearranged, generates a 15/13 with the subtonic.

H10. Characteristic Interval 50/39 430 cents

38	52/51 · 51/50 · 50/39	34 + 35 + 430
39	39/38 · 76/75 · 50/39	45 + 23 + 430
40	78/77 · 77/75 · 50/39	22 + 46 + 430

The pyknon is 26/25 (68 cents) and is inspired by Kathleen Schlesinger's (1939, 214) enharmonic Lydian harmonia.

H11. Characteristic Interval 32/25 427 cents

41	50/49 · 49/48 · 32/25	35 + 36 + 427	
42	75/73 · 73/72 · 32/25	46 + 24 + 427	
43	75/74 · 37/36 · 32/25	23 + 47 + 427	

This genus divides the 25/24 minor semitone (71 cents). The 32/25 is the 3/2's complement of 75/64, the 5-limit augmented second (5/4 · 5/4 · 5/4 · 3/2, reduced to one octave).

ENHARMONIC TETRACHORDS

E1. Characteristic Interval 23/18 424 cents

44	48/47 · 47/46 · 23/18	36 + 37 + 424	SCHLESINGER
45	36/35 · 70/69 · 23/18	49 + 25 + 424	WILSON
46	72/71 · 71/69 · 23/18	24 + 50 + 424	
47	30/29 · 116/115 · 23/18	59 + 15 + 424	WILSON
48	60/59 · 118/115 · 23/18	29 + 45 + 424	

This genus divides the 24/23 (74 cents) and lies on the boundary between the enharmonic and hyperenharmonic genera. It is analogous to the 9/7 genus but divides the hemiolic chromatic rather than the soft or intense diesis. Numbers 45 and 47 are from Wilson. Number 44 (Schlesinger 1939, 214) is the lower tetrachord of her enharmonic Phrygian harmonia.

E2. Characteristic Interval 88/69 421 cents

49	46/45 · 45/44 · 88/69	38 + 39 + 421	
50	69/67 · 67/66 · 88/69	51 + 26 + 421	
51	69/68 · 34/33 · 88/69	25 + 52 + 421	

The pyknon of this enharmonic genus is 23/22 (77 cents).

E3. Characteristic Interval 50/41 421 cents

52	320/313 · 313/306 · 51/40	38 + 39 + 421	
53	480/473 · 473/459 · 51/40	25 + 52 + 421	
54	240/233 · 466/459 · 51/40	51 + 26 + 421	

The pyknon is 160/153 (77 cents). The 51/40 is the 3/2's complement of 20/17.

E4. Characteristic Interval 14/11 418 cents

55	44/43 · 43/42 · 14/11	40 + 41 + 418	
56	33/32 · 64/63 · 14/11	53 + 27 + 418	
57	66/65 · 65/63 · 14/11	26 + 54 + 418	
58	88/87 · 29/28 · 14/11	20 + 61 + 418	
59	36/35 · 55/54 · 14/11	49 + 32 + 418	

60	50/49 · 77/75 · 14/11	35 + 46 + 418	
61	14/11 · 143/140 · 40/39	418 + 37 + 44	

This is a new genus whose pyknon is 22/21 (81 cents). The 14/11 is a supramajor third found in the harmonic series between the fourteenth and eleventh partials. It occurs in the Partch diamond and other extended systems of just intonation.

E5. Characteristic Interval 80/63 414 cents

62	42/41 · 41/40 · 80/63	42 + 42 + 414	
63	63/61 · 61/60 · 80/63	56 + 28 + 414	
64	63/62 · 31/30 · 80/63	27 + 57 + 414	

The pyknon of this enharmonic genus is 21/20 (84 cents), a common interval in septimal just intonation.

E6. Characteristic Interval 33/26 413 cents

65	208/203 · 203/198 · 33/26	42 + 43 + 413	
66	312/307 · 307/297 · 33/26	28 + 57 + 413	
67	312/302 · 302/297 · 33/26	56 + 29 + 413	
68	52/51 · 34/33 · 33/26	34 + 52 + 413	
69	26/25 · 100/99 · 33/26	68 + 18 + 413	
70	78/77 · 28/27 · 33/26	22 + 63 + 413	

The characteristic interval of this genus is the 3/2's complement of 13/11 and derives from the 22:26:33 triad. The pyknon is 104/99 (85 cents).

E7. Characteristic Interval 19/15 409 cents

71	40/39 · 39/38 · 19/15	44 + 45 + 409	ERATOSTHENES
72	30/29 · 58/57 · 19/15	59 + 30 + 409	
73	60/59 · 59/57 · 19/15	29 + 60 + 409	
74	28/27 · 135/133 · 19/15	63 + 26 + 409	

The pyknon, 20/19 (89 cents), of this historically important genus is very close to the Pythagorean limma, 256/243. Number 71 is a good approximation to Aristoxenos's enharmonic of 3 + 3 + 24 "parts," and, in fact, is both Eratosthenes's enharmonic tuning and Ptolemy's misinterpretation of Aristoxenos's geometric scheme (Wallis 1682, 170). The next two entries are 2:1 and 1:2 divisions of the pyknon in analogy with the usual Ptolemaic and later Islamic practices. Number 73 is a hypothetical Ptolemaic interpretation of a (pseudo-)Aristoxenian 2 + 4 + 24 parts. An echo of this genus may appear as the sub-40 division found on the fingerboard of the Tanbur of Baghdad, a stringed instrument (Helmholtz [1877] 1954, 517).

The last species is an analog of Archytas's enharmonic and the first makes a 15/13 with the subtonic.

E8. Characteristic Interval 81/64 408 cents

75	512/499 · 499/486 · 81/64	45 + 46 + 408	BOETHIUS
76	384/371 · 742/729 · 81/64	60 + 31 + 408	
77	768/755 · 755/729 · 81/64	30 + 61 + 408	
78	40/39 · 416/405 · 81/64	44 + 46 + 408	
79	128/125 · 250/243 · 81/64	41 + 49 + 408	EULER
80	64/63 · 28/27 · 81/64	27 + 63 + 408	WILSON
81	$3^{24}/2^{38} \cdot 2^{46}/3^{29} \cdot 81/64$	47 + 43 + 408	
82	36/35 · 2240/2187 · 81/64	49 + 41 + 408	

In these tunings the limma, 256/243 (90 cents), has been divided. Number 75 is the enharmonic of Boethius and is obtained by a simple linear division of the pyknon. It represents Aristoxenos's enharmonic quite well, but see the preceding 19/15 genera for a solution more convenient on the monochord. In practice, the two (numbers 71 and 75) could not be distinguished by ear. Numbers 76 and 77 are triple divisions of the pyknon, for which Wilson's division is a convenient and harmonious approximation. Number 78 is an approximation to number 75, as is Euler's "old enharmonic" (Euler [1739] 1960, 170). Wilson's tuning (number 80) should also be compared to the Serre division of the 16/15 (5/4 genus). When number 80 is rearranged, the 28/27 will make a 7/6 with the subtonics hyperhypate or mese. In this form, it is a possible model for a tuning transitional between Aristoxenos's and Archytas's enharmonics. The purely Pythagorean division (number 81) is obtained by tuning five fifths down for the limma and twenty-four up for the double comma. Number 82 is found in Vogel's tuning (1963, 1967) and resembles Euler's (number 79).

E9. Characteristic Interval 24/19 404 cents

83	38/37 · 37/36 · 24/19	46 + 47 + 404	
84	57/55 · 55/54 · 24/19	62 + 32 + 404	
85	57/56 · 28/27 · 24/19	31 + 63 + 404	WILSON
86	76/75 · 25/24 · 24/19	23 + 71 + 404	
87	40/39 · 117/95 · 24/19	44 + 50 + 404	

The pyknon is 19/18 (94 cents). The interval of 24/19 derives from the 16:19:24 minor triad, which Shirlaw attributes to Ousley (Shirlaw 1917, 434) and which generates the corresponding tritriadic scale. It is the 3/2 complement of 19/16.

E10. Characteristic Interval 34/27 399 cents

88	36/35 · 35/34 · 34/27	49 + 50 + 399
89	27/26 · 52/51 · 34/27	65 + 34 + 399
90	54/53 · 53/51 · 34/27	32 + 67 + 399
91	24/23 · 69/68 · 34/27	74 + 25 + 399

This genus divides the 18/17 semitone of 99 cents, used by Vincenzo Galilei in his lute fretting (Barbour 1953; Lindley 1984). These genera are virtually equally-tempered and number 88 is an excellent approximation to Aristoxenos's enharmonic. It is also the first trichromatic of Schlesinger's Phrygian harmonia.

E11. Characteristic Interval 113/90 394 cents

92	240/233 · 233/226 · 113/90	51 + 53 + 394
93	180/173 · 346/339 · 113/90	69 + 35 + 394
94	360/353 · 353/339 · 113/90	34 + 70 + 394
95	30/29 · 116/113 · 113/90	59 + 45 + 394
96	40/39 · 117/113 · 113/90	44 + 60 + 394
97	60/59 · 118/113 · 113/90	29 + 75 + 394

These complex divisions derive from an attempt to interpret in Ptolemaic terms a hypothetical Aristoxenian genus of 7 + 23 parts. The inspiration came from Winnington-Ingram's 1932 article on Aristoxenos in which he discusses Archytas's 28/27 · 36/35 · 5/4 enharmonic genus and its absence from Aristoxenos's genera, despite the somewhat grudging acceptance of Archytas's other divisions. In Aristoxenian terms, Archytas's enharmonic would be 4 + 3 + 23 parts, and the first division is 3.5 + 3.5 + 23. Number 95 is the 4 + 3 division and 93 and 94 are 2:1 and 1:2 divisions of the complex pyknon of ratio 120/113 (104 cents). Numbers 96 and 97 are simplifications, while number 96 generates an ekbole of 5 dieses (15/13) with the subtonics hyperhypate and mese.

E12. Characteristic Interval 64/51 393 cents

98	34/33 · 33/32 · 64/51	52 + 53 + 393
99	51/50 · 25/24 · 64/51	34 + 71 + 393
100	49/48 · 51/49 · 64/51	36 + 69 + 393
101	68/65 · 65/64 · 64/51	78 + 27 + 393
102	68/67 · 67/64 · 64/51	26 + 79 + 393

The pyknon of this enharmonic genus is 17/16 (105 cents), the seventeenth harmonic and a basic interval in *septendecimal* just intonation.

E13. CHARACTERISTIC INTERVAL 5/4 386 CENTS

103	32/31 · 31/30 · 5/4	55 + 57 + 386	DIDYMOS
104	46/45 · 24/23 · 5/4	38 + 74 + 386	PTOLEMY
105	48/47 · 47/45 · 5/4	36 + 75 + 386	
106	28/27 · 36/35 · 5/4	63 + 49 + 386	ARCHYTAS
107	56/55 · 22/21 · 5/4	31 + 81 + 386	PTOLEMY?
108	40/39 · 26/25 · 5/4	44 + 68 + 386	AVICENNA
109	25/24 · 128/125 · 5/4	71 + 41 + 386	SALINAS
110	21/20 · 64/63 · 5/4	84 + 27 + 386	PACHYMERES
111	256/243 · 81/80 · 5/4	90 + 22 + 386	FOX-STRANGWAYS?
112	76/75 · 20/19 · 5/4	23 + 89 + 386	
113	96/95 · 19/18 · 5/4	18 + 94 + 386	WILSON
114	136/135 · 18/17 · 5/4	13 + 99 + 386	HOFMANN
115	256/255 · 17/16 · 5/4	7 + 105 + 386	HOFMANN
116	68/65 · 5/4 · 52/51	78 + 386 + 34	

These tunings are the most consonant of the shades of the enharmonic genera. Although Plato alludes to the enharmonic, the oldest tuning we actually have is that of Archytas (390 BCE). This tuning, number 106, clearly formed part of a larger musical system which included the subtonic and the tetrachord synemmenon as well as both the diatonic and chromatic genera (Winnington-Ingram 1932; Erickson 1965). Didymos's tuning is the 1:1 division of the 16/15 (112 cents) pyknon and dates from a time when the enharmonic had fallen out of use. Number 104 is undoubtedly Ptolemy's own, but the surviving manuscripts contain an extra page which lists number 107 instead. Wallis believed it to be a later addition, probably correctly. Numbers 104 and 105 are the 1:2 and 2:1 divisions, given as usual for illustrative and/or pedagogical purposes. The Avicenna tuning (D'Erlanger 1935, 154) has the 5/4 first in the original, following the usual practice of the Islamic theorists. In this form, it makes a 15/13 with the subtonic. Number 109 is Euler's enharmonic (Euler [1739] 1960, 178); Hawkins, however, attributes it to Salinas (Hawkins [1776] 1963, 27). Daniélou gives it in an approximation with 46/45 replacing the correct 128/125 (Daniélou 1943, 175). The Pachymeres enharmonic is attributed by Perrett to Tartini (Perrett 1926, 26), but Bryennios and Serre also list it.

Number 111 is given as *Rag Todi* by Fox-Strangways (1916, 121) and as *Gunakali* by Daniélou (1959, 134–135). The divisions with extraordinarily small intervals, numbers 114 and 115, were found by Hofmann in his

computation of the 26 possible superparticular divisions of the 4/3 (Vogel 1975).

E14. Characteristic Interval 8192/6561 384 cents

117	4374/4235 · 4235/4096 · 8192/6561	57 + 57 + 384	
118	6561/6283 · 6283/6144 · 8192/6561	75 + 39 + 384	
119	6561/6422 · 3211/3072 · 8192/6561	37 + 77 + 384	
120	$3^{24}/2^{38} \cdot 2^{27}/3^{17} \cdot 8192/6561$	47 + 68 + 384	

The interval 8192/6561 is Helmholtz's *skhismic* major third, which is generated by tuning eight fifths down and five octaves up (Helmholtz [1877] 1954, 432). The pyknon is the apotome, 2187/2048 (114 cents). It has been linearly divided in the first three tetrachords above, but a purely Pythagorean division is given as number 120.

E15. Characteristic Interval 56/45 379 cents

121	30/29 · 29/28 · 56/45	59 + 60 + 379	PTOLEMY
122	45/43 · 43/42 · 56/45	79 + 41 + 379	
123	45/44 · 22/21 · 56/45	39 + 53 + 379	
124	25/24 · 36/35 · 56/45	71 + 49 + 379	
125	80/77 · 33/32 · 56/45	66 + 53 + 379	
126	60/59 · 59/56 · 56/45	29 + 90 + 379	
127	40/39 · 117/112 · 56/45	44 + 76 + 379	
128	26/25 · 375/364 · 56/45	68 + 52 + 379	

The pyknon is 15/14 (119 cents). Number 121 is Ptolemy's interpretation of Aristoxenos's soft chromatic, 4 + 4 + 22 parts. Number 125 is a Ptolemaic interpretation of a hypothetical 4.5 + 3.5 + 22 parts, an approximation to Archytas's enharmonic (Winnington-Ingram 1932). Number 124 is a simplification of the former tuning, and numbers 122 and 123 are the familiar threefold divisions. Number 128 is a summation tetrachord.

E16. Characteristic Interval 41/33 376 cents

129	88/85 · 85/82 · 41/33	60 + 62 + 376	
130	42/41 · 22/21 · 41/33	42 + 81 + 376	
131	44/43 · 43/41 · 41/43	39 + 82 + 376	

This genus is an attempt to approximate a theoretical genus, 62.5 + 62.5 + 375 cents, which would lie on the border between the chromatic and enharmonic genera. Number 129 is quite close, and numbers 130 and 131 are 1:2 and 2:1 divisions of the complex 44/41 (122 cents) pyknon.

CHROMATIC TETRACHORDS

C1. Characteristic Interval 36/29 374 cents

132	29/28 · 28/27 · 36/29	61 + 63 + 374	
133	87/85 · 85/81 · 36/29	40 + 83 + 374	
134	87/83 · 83/81 · 36/29	81 + 42 + 374	

This genus is also an approximation to 62.5 + 62.5 + 375 cents. The 36/29 is from the 24:29:36 triad and tritriadic scale. The pyknon is 29/27 (124 cents).

C2. Characteristic Interval 26/21 370 cents

135	28/27 · 27/26 · 26/21	63 + 65 + 370	SCHLESINGER
136	21/20 · 40/39 · 26/21	85 + 44 + 370	
137	42/41 · 41/39 · 26/21	42 + 87 + 370	
138	24/23 · 161/156 · 26/21	74 + 55 + 370	

This genus divides the pyknon, 14/13 (128 cents) and approximates Aristoxenos's soft chromatic. Number 135 is from Schlesinger (1933) and is a first tetrachord of a modified Mixolydian harmonia.

C3. Characteristic Interval 21/17 366 cents

139	136/131 · 131/126 · 21/17	65 + 67 + 366	
140	102/97 · 194/189 · 21/17	87 + 45 + 366	
141	204/199 · 199/189 · 21/17	43 + 89 + 366	
142	64/63 · 17/16 · 21/17	27 + 105 + 366	
143	34/33 · 22/21 · 21/17	52 + 81 + 366	
144	40/39 · 221/210 · 21/17	44 + 88 + 366	
145	24/23 · 391/378 · 21/17	74 + 59 + 366	
146	28/27 · 51/49 · 21/17	63 + 69 + 366	

The pyknon is 68/63 (132 cents). Number 139 is a very close approximation of Aristoxenos's soft chromatic, 4 + 4 + 22 "parts," as is number 146 also. Numbers 144 and 146 make intervals of 15/13 and 7/6, respectively, with their subtonics.

C4. Characteristic Interval 100/81 365 cents

147	27/26 · 26/25 · 100/81	65 + 68 + 365	
148	81/77 · 77/75 · 100/81	87 + 46 + 365	
149	81/79 · 79/75 · 100/81	45 + 88 + 365	
150	81/80 · 16/15 · 100/81	22 + 112 + 365	
151	51/50 · 18/17 · 100/81	34 + 99 + 365	
152	36/35 · 21/20 · 100/81	49 + 85 + 365	

153	40/39 · 1053/1000 · 100/81	44 + 89 + 365	
154	135/128 · 128/125 · 100/81	92 + 41 + 365	DANIÉLOU
155	24/23 · 207/200 · 100/81	74 + 60 + 365	

The pyknon is the great limma or large chromatic semitone, 27/25 (133 cents). Daniélou listed his tetrachord in approximate form with 46/45 instead of the correct 128/125. (Daniélou 1943, 175). Number 147 is a close approximation to Aristoxenos's soft chromatic, but the rest of the divisions are rather complex.

C5. Characteristic Interval 37/30 363 cents

156	80/77 · 77/74 · 37/30	66 + 69 + 363	PTOLEMY
157	20/19 · 38/37 · 37/30	89 + 46 + 363	
158	40/39 · 39/37 · 37/30	44 + 91 + 363	
159	30/29 · 116/111 · 37/30	59 + 76 + 363	
160	60/59 · 118/111 · 37/30	29 + 106 + 363	

This complex chromatic genus divides the 40/37 (135 cents). Number 156 is Ptolemy's linear interpretation of Aristoxenos's hemiolic chromatic, 4.5 + 4.5 + 21 "parts," with its characteristic neutral third and 3/4-tone pyknon. This division closely approximates his soft chromatic, indicating that Ptolemy's interpretation in terms of the aliquot parts of a real string was erroneous and that Aristoxenos really did mean something conceptually similar to equal temperament. However, Ptolemy's approach and the resulting tetrachords are often interesting in their own right. For example, number 157 could be considered as a Ptolemaic version of Aristoxenos's 1/2 + 1/4 + 1 3/4 tones, 6 + 3 + 21 "parts," a genus rejected as unmelodic because the second interval is smaller than the first (Winnington-Ingram 1932). The remaining genera are experimental.

C6. Characteristic Interval 16/13 359 cents

161	26/25 · 25/24 · 16/13	68 + 71 + 359
162	39/37 · 37/36 · 16/13	91 + 47 + 359
163	39/38 · 19/18 · 16/13	45 + 94 + 359
164	65/64 · 16/15 · 16/13	27 + 112 + 359
165	52/51 · 17/16 · 16/13	34 + 105 + 359
166	40/39 · 169/160 · 16/13	44 + 95 + 359
167	28/27 · 117/112 · 16/13	63 + 76 + 359
168	169/168 · 14/13 · 16/13	11 + 128 + 359
169	22/21 · 91/88 · 16/13	81 + 58 + 359

The pyknon of this genus, which lies between the soft and hemiolic

chromatics of Aristoxenos, is 13/12 (139 cents). Number 169 is a summation tetrachord from chapter 4.

C7. Characteristic Interval 27/22 355 cents

170	176/169 · 169/162 · 27/22	70 + 73 + 355	
171	132/125 · 250/243 · 27/22	94 + 49 + 355	
172	264/257 · 257/243 · 27/22	47 + 97 + 355	
173	28/27 · 22/21 · 27/22	63 + 81 + 355	
174	55/54 · 16/15 · 27/22	32 + 112 + 355	
175	40/39 · 143/135 · 27/22	44 + 100 + 355	

The *Wosta of Zalzal*, a neutral third of 355 cents, is exploited in this hemiolic chromatic genus whose pyknon is 88/81 (143 cents), an interval found in certain Islamic scales (D'Erlanger 1935).

C8. Characteristic Interval 11/9 347 cents

176	24/23 · 23/22 · 11/9	74 + 77 + 347	WINNINGTON-INGRAM
177	18/17 · 34/33 · 11/9	99 + 52 + 347	
178	36/35 · 35/33 · 11/9	49 + 102 + 347	
179	45/44 · 16/15 · 11/9	39 + 112 + 347	
180	56/55 · 15/14 · 11/9	31 + 119 + 347	
181	78/77 · 14/13 · 11/9	22 + 128 + 347	
182	20/19 · 57/55 · 11/9	89 + 62 + 347	
183	30/29 · 58/55 · 11/9	59 + 92 + 347	
184	28/27 · 81/77 · 11/9	63 + 88 + 347	
185	40/39 · 117/110 · 11/9	44 + 107 + 347	

This genus is the simplest realization of Aristoxenos's hemiolic chromatic. Winnington-Ingram mentions number 176 in his 1932 article on Aristoxenos but rejects it, despite using 12/11 · 11/9 to construct his spondeion scale in an earlier paper (Winnington-Ingram 1928). In view of the widespread use of 3/4-tone and neutral third intervals in extant Islamic music and the use of 12/11 by Ptolemy in his intense chromatic and equable diatonic genera, I see no problems with accepting Aristoxenos's genus, 4.5 + 4.5 + 21 "parts," as recording an actual tuning, traces of which are still to be found in the Near East. Ptolemy, it should be remembered, claimed that the intense chromatic, 22/21 · 12/11 · 7/6, was used in popular lyra and kithara tunings (Wallis 1682, 84, 178, 208) and that his equable diatonic sounded rather foreign and rustic. Schlesinger identifies it with the first tetrachord of her chromatic Phrygian harmonia (Schlesinger 1933; Schlesinger 1939, 214). The pyknon of this chromatic genus is 12/11 (151 cents). Number 176 may

be written as 5 + 5 + 20 Ptolemaic "parts" (120 115 110 90), rather than the 4.5 + 4.5 + 21 of Aristoxenian theory. A number of other divisions are shown, including the usual 1:2 and 2:1, as well as the neo-Archytan 28/27 and 40/39 types.

C9. CHARACTERISTIC INTERVAL 39/32 342 CENTS

186	256/245 · 245/234 · 39/32	76 + 80 + 342	
187	384/373 · 373/351 · 39/32	50 + 105 + 342	
188	192/181 · 362/351 · 39/32	102 + 53 + 342	
189	64/63 · 14/13 · 39/32	27 + 128 + 342	

This genus employs the 3/2's complement of 16/13, the tridecimal neutral third, found in the 26:32:39 triad. The unusually complex pyknon is 128/117 (156 cents).

C10. CHARACTERISTIC INTERVAL 28/23 341 CENTS

190	23/22 · 22/21 · 28/23	76 + 81 + 341	WILSON
191	69/65 · 65/63 · 28/23	103 + 54 + 341	
192	69/67 · 67/63 · 28/23	51 + 107 + 341	
193	46/45 · 15/14 · 28/23	38 + 119 + 341	

This neutral third genus is from Wilson. The pyknon is 23/21 (157 cents).

C11. CHARACTERISTIC INTERVAL 17/14 336 CENTS

194	112/107 · 107/102 · 17/14	79 + 83 + 336
195	168/158 · 158/153 · 17/14	106 + 56 + 336
196	168/163 · 163/153 · 17/14	52 + 110 + 336
197	52/51 · 14/13 · 17/14	34 + 128 + 336
198	28/27 · 18/17 · 17/14	63 + 99 + 336
199	35/34 · 16/15 · 17/14	50 + 112 + 336
200	40/39 · 91/85 · 17/14	44 + 118 + 336
201	17/14 · 56/55 · 55/51	336 + 31 + 131
202	17/14 · 56/53 · 53/51	336 + 95 + 67

This chromatic genus uses Ellis's supraminor third, 17/14 (Helmholtz [1877] 1954, 455), which occurs in his septendecimal interpretation of the diminished seventh chord, 10:12:14:17. The pyknon is 56/51 (162 cents).

C12. CHARACTERISTIC INTERVAL 40/33 333 CENTS

203	22/21 · 21/20 · 40/33	81 + 85 + 333
204	33/32 · 31/30 · 40/33	108 + 57 + 333
205	33/32 · 16/15 · 40/33	53 + 112 + 333
206	55/54 · 27/25 · 40/33	32 + 133 + 333

207	66/65 · 13/12 · 40/33	26 + 139 + 333	
208	18/17 · 187/180 · 40/33	99 + 66 + 333	

The pyknon of this genus is 11/10 (165 cents), an interval which appears in Ptolemy's equable diatonic and elsewhere. Number 208 is a summation tetrachord from chapter 4.

C13. CHARACTERISTIC INTERVAL 29/24 328 CENTS

209	64/61 · 61/58 · 29/24	83 + 87 + 328	
210	16/15 · 30/29 · 29/24	112 + 59 + 328	SCHLESINGER
211	32/31 · 31/29 · 29/24	55 + 115 + 328	SCHLESINGER

The interval 29/24 is found in some of Schlesinger's harmoniai when she tries to correlate her theory of linearly divided octaves with Greek notation (Schlesinger 1939, 527–8). The results agree neither with the commonly accepted interpretation of the notation, nor with the canonical forms of the harmoniai given elsewhere in her book. The 29/24 is also part of the 24:29:36 triad and its 3/2's complement generates the 36/29 genus. The pyknon is 32/29 (170 cents).

C14. CHARACTERISTIC INTERVAL 6/5 316 CENTS

212	20/19 · 19/18 · 6/5	89 + 94 + 316	ERATOSTHENES
213	28/27 · 15/14 · 6/5	63 + 119 + 316	PTOLEMY
214	30/29 · 29/27 · 6/5	59 + 123 + 316	
215	16/15 · 25/24 · 6/5	112 + 71 + 316	DIDYMOS
216	40/39 · 13/12 · 6/5	44 + 139 + 316	BARBOUR
217	55/54 · 12/11 · 6/5	32 + 151 + 316	BARBOUR
218	65/63 · 14/13 · 6/5	54 + 128 + 316	
219	22/21 · 35/33 · 6/5	81 + 102 + 316	
220	21/20 · 200/189 · 6/5	85 + 97 + 316	PERRETT
221	256/243 · 6/5 · 135/128	90 + 316 + 92	XENAKIS
222	60/59 · 59/54 · 6/5	29 + 153 + 316	
223	80/77 · 77/72 · 6/5	66 + 116 + 316	
224	24/23 · 115/108 · 6/5	74 + 109 + 316	
225	88/81 · 45/44 · 6/5	143 + 39 + 316	
226	46/45 · 6/5 · 25/23	38 + 316 + 144	
227	52/51 · 85/78 · 6/5	34 + 149 + 316	WILSON
228	100/99 · 11/10 · 6/5	17 + 165 + 316	HOFMANN
229	34/33 · 6/5 · 55/51	52 + 316 + 131	
230	6/5 · 35/32 · 64/63	316 + 155 + 27	
231	6/5 · 2240/2187 · 243/224	316 + 41 + 141	

This genus is the most consonant of the chromatic genera. Number 212 is the chromatic of Eratosthenes and is identical to Ptolemy's interpretation of Aristoxenos's intense chromatic genus. It is likely, however, that Aristoxenos's genus corresponds to one of the 32/27 genera. Number 213 is Ptolemy's soft chromatic and is the 2:1 division reordered. Number 214 is the 1:2 division and a Ptolemaic interpretation of a 4 + 8 + 18 "parts." Didymos's tuning is probably the most consonant, although it violates the usual melodic canon of Greek theory that the smallest interval must be at the bottom of the tetrachord. In reverse order, this tuning is produced by the seventh of Proclus's ten means (Heath 1921). Archytas's enharmonic and diatonic tunings also violate this rule; the rule may either be later or an ideal theoretical principle. Numbers 216 and 217 are from Barbour (1951, 23). Perrett's tetrachord, like one of the 25/21 genera, is found to occur unexpectedly in his new scale (Perrett 1926, 79). The Xenakis tetrachord (number 221) is from the article, "Towards a Metamusic," which has appeared in different translations in different places (Xenakis 1971). It also appears in Archytas's system according to Erickson (1965). The Hofmann genus is from Vogel (1975). Numbers 230 and 231 are found in Vogel's tuning (1963, 1967) and chapter 6. The pyknon is the minor tone 10/9 (182 cents).

C15. CHARACTERISTIC INTERVAL 25/21 302 CENTS

232	56/53 · 53/50 · 25/21	97 + 99 + 302	
233	14/13 · 26/25 · 25/21	128 + 68 + 302	
234	28/27 · 27/25 · 25/21	63 + 133 + 302	
235	21/20 · 16/15 · 25/21	84 + 112 + 302	PERRETT
236	40/39 · 273/250 · 25/21	44 + 152 + 302	

This genus whose pyknon is 28/25 (196 cents) is inspired by number 235, a tetrachord from Perrett (1926, 80). Number 232 is virtually equally tempered and number 234 is an excellent approximation to Aristoxenos's 1/3 + 2/3 + 1 1/2 tones, 4 + 8 + 18 "parts."

C16. CHARACTERISTIC INTERVAL 19/16 298 CENTS

237	128/121 · 121/114 · 19/16	97 + 103 + 298	
238	96/89 · 178/171 · 19/16	131 + 69 + 298	
239	192/185 · 185/171 · 19/16	64 + 136 + 298	
240	20/19 · 19/16 · 16/15	89 + 298 + 112	KORNERUP
241	256/243 · 81/76 · 19/16	90 + 110 + 298	BOETHIUS
242	96/95 · 10/9 · 19/16	18 + 182 + 298	WILSON

243	64/63 · 21/19 · 19/16	27 + 173 + 298	
244	40/39 · 104/95 · 19/16	44 + 157 + 298	

The characteristic ratio for this genus derives from the 16:19:24 minor triad (see the 24/19 genus). The pyknon is the complex interval 64/57 (201 cents). Number 241 is from Boethius (1838, 6). The Kornerup tetrachord (1934, 10) also corresponds to a Ptolemaic interpretation of one of Athanasopoulos's (1950) Byzantine tunings, 6 + 18 + 6 "parts." As 19/16 · 20/19 · 16/15, it is one of the "mean" tetrachords.

C17. Characteristic Interval 32/27 294 cents

245	18/17 · 17/16 · 32/27	99 + 105 + 294	ARISTIDES QUINT.
246	27/25 · 25/24 · 32/27	133 + 71 + 294	
247	27/26 · 13/12 · 32/27	65 + 139 + 294	BARBOUR?
248	28/27 · 243/224 · 32/27	63 + 141 + 294	ARCHYTAS
249	256/243 · 2187/2048 · 32/27	90 +114 + 294	GAUDENTIUS
250	81/80 · 10/9 · 32/27	22 + 182 + 294	BARBOUR?
251	33/32 · 12/11 · 32/27	53 + 151 + 294	BARBOUR?
252	45/44 + 11/10 · 32/27	39 + 165 + 294	BARBOUR?
253	21/20 · 15/14 · 32/27	84 + 119 + 294	PERRETT
254	135/128 · 16/15 · 32/27	92 + 112 + 294	
255	36/35 · 35/32 · 32/27	49 + 155 + 294	WILSON
256	49/48 · 54/49 · 32/27	36 + 168 + 294	WILSON
257	243/230 · 230/216 · 32/27	95 + 109 + 294	PS.-PHILOLAUS?
258	243/229 · 229/216 · 32/27	103 +101+ 294	
259	20/19 · 171/160 · 32/27	89 + 115 + 294	
260	23/22 · 99/92 · 32/27	77 + 127 + 294	
261	24/23 · 69/64 · 32/27	74 + 130 + 294	
262	40/39 + 351/320 · 32/27	44 + 160 + 294	
263	14/13 · 117/112 · 32/27	128 + 76 + 294	

These chromatic genera are derived from the traditional "Pythagorean" tuning (perfect fourths, fifths, and octaves), which is actually of Sumero-Babylonian origin (Duchesne-Guillemin 1963, 1969; Kilmer 1960), by changing the pitch of the second string, the parhypate or trite. Number 245, the 1:1 division of the 9/8 pyknon (204 cents), is from from the late classical writer, Aristides Quintilianus (Meibomius 1652, 123). Tunings numbers 246 and 254 are of obscure origin. They were constructed after reading a passage in Hawkins ([1776] 1963, 37) which quotes Wallis as crediting Mersenne with the discovery of the 27/25 and 135/128 semitones

and their 9/8 complements. However, the discussion is about diatonic genera, not chromatic, and it is unclear to me whether Mersenne really did construct these two chromatic tetrachords. Archytas's chromatic, number 248, has been identified with Aristoxenos's 1/3 + 2/3 + 1 1/2 tones by Winnington-Ingram (1932) and number 247 is a good approximation to his 1/2 + 1/2 + 1 1/2 tones. Number 249 is the unaltered Pythagorean version from Gaudentius. The Barbour tetrachords derive from his discussion of different superparticular divisions of the 9/8 (Barbour 1951, 154–156). Although tetrachords are mentioned, it is not clear that he ever actually constructed these divisions. Perrett discovered number 253, like number 235 above, in his scale after it was constructed. Both Chaignet (1874, 231) and McClain (1978, 160) quote (*Ps.*)-Philolaus as dividing the tone into 27 parts, 13 of which go to the minor semitone, and 14 to the major. Number 257 is the result of this division and number 258 has the parts taken in reverse order. It would seem that number 245 and number 258 are essentially equivalent to Aristoxenos's theoretical intense chromatic and that numbers 254, 257, 259, and probably 253 as well, are equivalent to Gaudentius's Pythagorean tuning. The presence of secondary ratios of 5 and 7 in number 253 and number 254 suggests that the equivalences would be melodic rather than harmonic. The last tuning is a summation tetrachord from chapter 4.

C18. Characteristic Interval 45/38 293 cents

264	304/287 · 287/270 · 45/38	100 + 106 + 293
265	456/439 · 439/405 · 45/38	66 + 140 + 293
266	228/211 · 422/405 · 45/38	134 + 71 + 293
267	19/18 · 16/15 · 45/38	94 + 112 + 293
268	76/75 · 10/9 · 45/38	23 + 182 + 293
269	38/35 · 28/27 · 45/38	142 + 63 + 293

This genus uses the 45/38, the 3/2's complement of 19/15. The pyknon is 152/135 (205 cents). Number 264 is a reasonable approximation to the intense chromatic and number 269 is similar to Archytas's chromatic, if rearranged with the 28/27 first.

C19. Characteristic Interval 13/11 289 cents

270	88/83 · 83/78 · 13/11	101 + 108 + 289
271	66/61 · 122/117 · 13/11	136 + 72 + 289
272	132/127 · 127/117 · 13/11	67 + 142 + 289
273	14/13 · 22/21 · 13/11	128 + 81 + 289
274	40/39 · 11/10 · 13/11	44 + 165 + 289

275	66/65 · 10/9 · 13/11	26 + 182 + 289	WILSON
276	27/26 · 88/81 · 13/11	65 + 143 + 289	
277	28/27 · 99/91 · 13/11	63 + 146 + 289	

This experimental genus divides a pyknon of 44/39 (209 cents), an interval also appearing in William Lyman Young's diatonic lyre tuning (Young 1961). The 13/11 is a minor third which appears in 13-limit tunings and with its 3/2's complement, 33/26, generates the 22:26:33 tritriadic scale.

C20. Characteristic Interval 33/28 284 cents

278	224/211 · 211/198 · 33/28	104 + 110 + 284
279	336/323 · 323/297 · 33/28	68 + 145 + 284
280	168/155 · 310/297 · 33/28	139 + 74 + 284
281	56/55 · 10/9 · 33/28	31 + 182 + 284
282	16/15 · 35/32 · 33/28	112 + 102 + 284
283	34/33 · 33/28 · 56/51	52 + 284 + 162

The characteristic interval of this genus is the 3/2's complement of 14/11, 33/28. The pyknon is 112/99 (214 cents).

C21. Characteristic Interval 20/17 281 cents

284	17/16 · 16/15 · 20/17	105 + 112 + 281
285	51/47 · 47/45 · 20/17	142 + 75 + 281
286	51/49 · 49/45 · 20/17	69 + 147 + 281
287	34/33 · 11/10 · 20/17	52 + 165 + 281
288	51/50 · 10/9 · 20/17	34 + 182 + 281
289	40/39 · 221/200 · 20/17	44 + 173 + 281
290	28/27 · 153/140 · 20/17	63 + 154 + 281
291	21/20 · 20/17 · 68/63	85 + 281 + 132
292	68/65 · 13/12 · 20/17	78 + 139 + 281
293	34/31 · 31/30 · 20/17	160 + 57 + 281
294	68/61 · 61/60 · 20/17	188 + 29 + 281
295	68/67 · 67/57 · 19/17	26 + 280 + 193
296	68/67 · 67/60 · 20/17	26 + 191 + 281

The pyknon is 17/15 (217 cents). Intervals of 17 are becoming increasingly common in justly-intoned music. This would appear to be a metaphysical phenomenon of considerable philosophical interest (Polansky, personal communication).

C22. Characteristic Interval 27/23 278 cents

297	184/173 · 173/162 · 27/23	107 + 114 + 278
298	276/265 · 265/243 · 27/23	70 + 150 + 278

299	138/127 · 254/243 · 27/2	144 + 77 + 278	
300	28/27 · 23/21 · 27/23	63 + 157 + 278	
301	23/22 · 88/81 · 27/23	77 + 143 + 278	
302	46/45 · 10/9 · 27/23	38 + 182 + 278	

This genus exploits the 3/2's complement of 23/18, which is derived from the 18:23:27 triad. The pyknon is 92/81 (220 cents).

C23. CHARACTERISTIC INTERVAL 75/64 275 CENTS

303	512/481 · 481/450 · 75/64	108 + 115 + 275	
304	768/737 · 737/675 · 75/64	71 + 152 + 275	
305	384/353 · 706/675 · 75/64	146 + 78 + 275	
306	16/15 · 75/64 · 16/15	112 + 275 + 112	HELMHOLTZ

The pyknon is 256/225 (223 cents). The 75/64 is the 5-limit augmented second, which appears, for example, in the harmonic minor scale. Helmholtz's tetrachord is from (Helmholtz [1877] 1954, 263).

C24. CHARACTERISTIC INTERVAL 7/6 267 CENTS

307	16/15 · 15/14 · 7/6	112 + 119 + 267	AL-FARABI
308	22/21 · 12/11 · 7/6	81 + 151 + 267	PTOLEMY
309	24/23 · 23/21 · 7/6	74 + 157 + 267	
310	20/19 · 38/35 · 7/6	89 + 142 + 267	PTOLEMY
311	10/9 · 36/35 · 7/6	182 + 49 + 267	AVICENNA
312	64/63 · 9/8 · 7/6	27 + 204 + 267	BARBOUR
313	92/91 · 26/23 · 7/6	19 + 212 + 267	
314	256/243 · 243/224 · 7/6	90 + 141 + 267	HIPKINS
315	40/39 · 39/35 · 7/6	44 + 187 + 267	
316	18/17 · 7/6 · 68/63	99 + 267 + 132	
317	50/49 · 7/6 · 28/25	35 + 267 + 196	
318	14/13 · 7/6 · 52/49	128 + 267 + 103	
319	46/45 · 180/161 · 7/6	38 + 193 + 267	
320	28/27 · 54/49 · 7/6	63 + 168 + 267	
321	120/113 · 113/105 · 7/6	104 + 127 + 267	
322	60/59 · 118/105 · 7/6	29 + 202 + 267	
323	30/29 · 116/105 · 7/6	59 + 172 + 267	
324	88/81 · 81/77 · 7/6	143 + 88 + 267	
325	120/119 · 17/15 · 7/6	14 + 217 + 267	
326	27/25 · 7/6 · 200/189	133 + 267 + 98	
327	26/25 · 7/6 · 100/91	68 + 267 + 163	

328 7/6 · 1024/945 · 135/128 267 + 139 + 92

The pyknon of this intense chromatic is the septimal tone, 8/7 (231 cents). Number 307 is given by Al-Farabi (D'Erlanger 1930, 104) and by Sachs (1943, 282) in rearranged form as the lower tetrachord of the modern Islamic mode, *Higaz*. The Turkish mode, *Zirgule*, has also been reported to contain this tetrachord, also with the 7/6 medially (Palmer 1967?). Vincent attributes this division to the Byzantine theorist, Pachymeres (Vincent 1847). This tuning is also produced by the harmonic mean operation. Ptolemy's first division (number 308) is his intense chromatic (Wallis 1682, 172), and his second (number 310) is his interpretation of Aristoxenos's soft diatonic, 6 + 9 + 15 "parts". In this instance, Ptolemy is not too far from the canonical 100 + 150 + 250 cents, though Hipkins's semi-Pythagorean solution (number 314) is more realistic (Vogel 1963). His tuning is also present in Erickson's (1965) interpretation of Archytas's system. The Avicenna tetrachord, number 311, (D'Erlanger 1935, 152) sounds, surprisingly, rather diatonic. Barbour's (1951, 23–24) tuning (number 312) is particularly attractive when arranged as 9/8 · 64/63 · 7/6. It also generates the 16:21:24 tritriadic and its conjugate. Vogel (1975, 207) lists it also. Number 328 is found in Vogel's tuning (chapter 6 and Vogel 1963, 1967). The remaining divisions are new tetrachords intended as variations on the soft diatonic-intense chromatic genus or as approximations of various Byzantine tetrachords as described by several authors (Xenakis 1971; Savas 1965; Athanasopoulos 1950).

C25. CHARACTERISTIC INTERVAL 136/117 261 CENTS

329 78/73 · 73/68 · 136/117 115 + 123 + 261

330 117/112 · 56/51 · 136/117 76 + 162 + 261

331 117/107 · 107/102 · 136/117 155 + 83 + 261

332 52/51 · 9/8 · 136/117 34 + 204 + 261

The pyknon of this complex genus is 39/34 (238 cents). Number 332 generates the 26:34:39 tritriadic.

C26. CHARACTERISTIC INTERVAL 36/31 259 CENTS

333 31/29 · 29/27 · 36/31 115 + 124 + 259

334 93/89 · 89/81 · 36/31 76 + 163 + 259

335 93/85 · 85/81 + 36/31 156 + 83 + 259

The pyknon is 31/27 (239 cents). The 36/31 is the 3/2's complement of 31/24, which defines a hyperenharmonic genus.

C27. Characteristic Interval 80/69 256 cents

336	46/43 · 43/40 · 80/69	117 + 125 + 256	
337	23/21 · 21/20 · 80/69	157 + 85 + 256	
338	23/22 · 11/10 · 80/69	77 + 165 + 256	
339	46/45 · 9/8 · 80/69	38 + 204 + 256	

The genus derives from number 339 which generates the 20:23:30 and 46:60:69 tritriadics. The pyknon is 23/20 (242 cents). This and the next few genera are realizations of Aristoxenos's soft diatonic.

C28. Characteristic Interval 22/19 254 cents

340	76/71 · 71/66 · 22/19	118 + 126 + 254	
341	57/52 · 104/99 · 22/19	159 + 85 + 254	
342	114/109 · 109/99 · 22/19	78 + 167 + 254	
343	19/18 · 12/11 · 22/19	94 + 151 + 254	SCHLESINGER
344.	34/33 · 19/17 · 22/19	52 + 192 + 254	
345	40/39 · 247/220 · 22/19	44 + 200 + 254	

This genus is a good approximation to the soft diatonic. Number 343 is from a folk scale (Schlesinger 1939, 297). Tetrachord numbers 344 and 345 are close to 3 + 12 + 15 "parts", a neo-Aristoxenian genus which mixes enharmonic and diatonic intervals. The pyknon is 38/33 (244 cents).

C29. Characteristic Interval 52/45 250 cents

346	15/14 · 14/13 · 52/45	119 + 128 + 250
347	45/41 · 41/39 · 52/45	161 + 87 + 250
348	45/43 · 43/39 · 52/45	78 + 169 + 250
349	24/23 · 115/104 · 52/45	74 + 174 + 250
350	40/39 · 9/8 · 52/45	44 + 204 + 250
351	18/17 · 85/78 · 52/45	99 + 149 + 250
352	45/44 · 44/39 · 52/45	39 + 209 + 250
353	65/63 · 28/25 · 52/45	54 + 196 + 250
354	55/52 · 12/11 · 52/45	97 + 151 + 250
355	60/59 · 59/45 · 52/45	29 + 219 + 250
356	20/19 · 52/45 · 57/52	89 + 250 + 149
357	27/26 · 10/9 · 52/45	66 + 182 + 250
358	11/10 · 150/143 · 52/45	165 + 83 + 250

This genus lies on the dividing line between the chromatic and diatonic genera. The pyknon of 15/13 (248 cents) is virtually identical to the CI which defines the genus. The first three subgenera are the 1:1, 2:1, and 1:2 divisions respectively. Number 350 generates the 10:13:15 tritriadic scale.

D1. CHARACTERISTIC INTERVAL 15/13 248 CENTS

359	104/97 · 97/90 · 15/13	124 + 126 + 248	
360	78/71 · 142/135 · 15/13	163 + 86 + 248	
361	156/149 · 149/135 · 15/13	79 + 171 + 248	
362	16/15 · 15/13 · 13/12	112 + 248 + 139	SCHLESINGER
363	26/25 · 10/9 · 15/13	68 + 182 + 248	
364	256/243 · 351/320 · 15/13	90 + 160 + 248	
365	20/19 · 247/225 · 15/13	89 + 161 + 248	
366	11/10 · 15/13 · 104/99	165 + 248 + 85	
367	12/11 · 15/13 · 143/135	151 + 248 + 99	
368	46/45 · 26/23 · 15/13	38 + 212 + 248	
369	40/39 · 169/150 · 15/13	44 + 206 + 248	
370	28/27 · 39/35 · 15/13	63 + 187 + 248	
371	91/90 · 8/7 · 15/13	19 + 231 + 248	

This genus is the first indubitably diatonic genus. A pyknon, *per se*, no longer exists because the 52/45 (250 cents) is larger than one-half the perfect fourth, 4/3 (498 cents). The large composite interval in this and succeeding genera is termed the "apyknon" or non-condensation (Bryennios). Number 362 is the first tetrachord of Schlesinger's diatonic Hypodorian harmonia. Many members of this genus are reasonable approximations to Aristoxenos's soft diatonic genus, 100 + 150 + 250 cents. Others with the 15/13 medially are similar to some Byzantine tunings. Some resemble the theoretical genus 50 + 200 + 250 cents.

D2. CHARACTERISTIC INTERVAL 38/23 244 CENTS

372	44/41 · 41/38 · 38/33	123 + 131 + 244
373	11/10 · 20/19 · 38/33	165 + 89 + 244
374	22/21 · 21/19 · 38/33	81 + 173 + 244

This genus divides the 22/19 (254 cents).

D3. CHARACTERISTIC INTERVAL 23/20 242 CENTS

375	160/149 · 149/138 · 23/20	123 + 133 + 242	
376	120/109 · 218/207 · 23/20	166 + 90 + 242	
377	240/229 · 229/207 · 23/20	81 + 175 + 242	
378	8/7 · 70/69 · 23/20	231 + 25 + 242	
379	40/39 · 26/23 · 23/20	44 + 212 + 242	
380	24/23 · 23/20 · 10/9	74 + 242 + 182	SCHLESINGER

381	28/27 · 180/161 · 23/20	63 + 193 + 242	

This genus is derived from the 20:23:30 triad. The apyknon is 80/69 (256 cents), Number 380 is from Schlesinger (1932) and is described as a harmonia of "artificial formula, Phrygian". Numbers 379 and 381 make intervals of 15/13 and 7/6 respectively with their subtonics. These intervals should be contrasted with the incomposite 23/20 in the tetrachord.

D4. Characteristic Interval 31/27 239 cents

382	72/67 · 67/62 · 31/27	125 + 134 + 239	
383	108/103 · 103/93 · 31/27	82 + 177 + 239	
384	54/49 · 98/93 · 31/27	168 + 91 + 239	
385	32/31 · 9/8 · 31/27	55 + 204 + 239	

The apyknon of this genus is 36/27 (259 cents). Number 385 generates the 24:31:36 tritriadic.

D5. Characteristic Interval 39/34 238 cents

386	272/253 · 253/234 · 39/34	125 + 135 + 238	
387	408/389 · 389/351 · 39/34	83 + 178 + 238	
388	204/185 · 370/351 · 39/34	169 + 91 + 238	
389	40/39 · 39/34 · 17/15	44 + 238 + 217	

The apyknon is 136/117 (261 cents). The 39/34 interval is the 3/2's complement of 17/13 and derives from the 26:34:39 triad.

D6. Characteristic Interval 8/7 231 cents

390	14/13 · 13/12 · 8/7	128 + 139 + 231	AVICENNA
391	19/18 · 21/19 · 8/7	94 + 173 + 231	SAFIYU-D-DIN
392	21/20 · 10/9 · 8/7	84 + 182 + 231	PTOLEMY
393	28/27 · 8/7 · 9/8	63 + 231 + 204	ARCHYTAS
394	49/48 · 8/7 · 8/7	36 + 231 + 231	AL-FARABI
395	35/33 · 11/10 · 8/7	102 + 165 + 231	AVICENNA
396	77/72 · 12/11 · 8/7	116 + 151 + 231	AVICENNA
397	16/15 · 35/32 · 8/7	112 + 155 + 231	VOGEL
398	35/34 · 17/15 · 8/7	50 + 217 + 231	
399	25/24 · 8/7 · 28/25	71 + 231 + 196	
400	15/14 · 8/7 · 49/45	119 + 231 + 147	
401	40/39 · 91/80 · 8/7	44 + 223 + 231	
402	46/45 · 105/92 · 8/7	38 + 229 + 231	
403	18/17 · 119/108 · 8/7	99 + 168 + 231	
404	17/16 · 8/7 · 56/51	105 + 231 + 162	
405	34/33 · 77/68 · 8/7	52 + 215 + 231	

406	256/243 · 567/512 · 8/7	90 + 177 + 231

This genus divides the 7/6 (267 cents). The Avicenna and Al-Farabi references are from D'Erlanger. Number 390 is also given by Pachymeres (D'Erlanger 1935, 148 referring to Vincent 1847). When arranged as 13/12 · 14/13 · 8/7, it is generated by taking two successive arithmetic means. Number 394 is especially interesting as there have been reports that it was used on organs in the Middle Ages (Adler 1968; Sachs 1949), but more recent work suggests that this opinion was due to a combination of transmission errors (by copyists) and an incorrect assessment of end correction (Barbour 1950; Munxelhaus 1976). With the 49/48 medially, it is generated by the twelfth of the Greek means (Heath 1921). The scale is obviously constructed in analogy with the Pythagorean 256/243 · 9/8 · 9/8. Similar claims pro and con have been made for number 393 as well. This scale, however, appears to have been the principal tuning of the diatonic in practice from the time of Archytas (390 BCE) through that of Ptolemy (ca. 160 CE). Even Aristoxenos grudgingly mentions it (Winnington-Ingram 1932). Number 397 is from Vogel (1963) and approximates the soft diatonic. It is also found in Erickson's (1965) version of Archytas's system. Entry 399 corresponds to 3/8 + 1 1/8 + 1 tones of Aristoxenos. The Safiyu-d-Din tuning is one of his "strong" forms (2:1 division) and has 21/19 replacing the 10/9 of Ptolemy. Tetrachords 403, 404, and 405 exploit ratios of 17 and are dedicated to Larry Polansky.

D7. CHARACTERISTIC INTERVAL 256/225 223 CENTS

407	150/139 · 139/128 · 256/225	132 + 143 + 223
408	225/214 · 107/96 · 256/225	87 + 188 + 223
409	225/203 · 203/192 · 256/225	78 + 96 + 223
410	25/24 · 9/8 · 256/225	71 + 204 + 223

The apyknon is the augmented second, 75/64 (275 cents). Number 410 is the generator of the 64:75:96 tritriadic and a good approximation to Aristoxenos's 3/8 + 1 1/8 + 1 tone when reordered so that the 9/8 is uppermost.

D8. CHARACTERISTIC INTERVAL 25/22 221 CENTS

411	176/163 · 163/150 · 25/22	133 + 144 + 221
412	132/119 · 238/225 · 25/22	179 + 97 + 221
413	264/251 · 251/225 · 25/22	87 + 189 + 221
414	16/15 · 11/10 · 25/22	112 + 165 + 221
415	88/81 · 27/25 · 25/22	143 + 133 + 221

416	22/21 · 25/22 · 28/25	81 + 221 + 196	
417	28/27 · 198/175 · 25/22	63 + 214 + 221	
418	26/25 · 44/39 · 25/22	68 + 209 + 221	

This is an experimental genus whose apyknon is 88/75 (277 cents). Number 416 is a fair approximation of Aristoxenos's 3/8 + 1 1/8 + 1 tones, and number 411 is close to a hypothetical 11/16 + 11/16 + 1 1/8 tones.

D9. Characteristic Interval 92/81 220 cents

419	27/25 · 25/23 · 92/81	133 + 144 + 220	
420	81/77 · 77/69 · 92/81	88 + 190 + 220	
421	81/73 · 73/69 · 92/81	180 + 98 + 220	
422	24/23 · 9/8 · 92/81	74 + 204 + 220	
423	27/26 · 26/23 · 92/81	66 + 212 + 220	

This genus divides the 27/23 (278 cents) and is derived from the 18:23:27 triad. Number 422 is the tritriadic generator, and is an approximation to Aristoxenos's 3/8 + 1 1/8 + 1 tones (4.5 + 13.5 + 12 "parts") when reordered.

D10. Characteristic Interval 76/67 218 cents

424	67/62 · 62/57 · 76/67	134 + 146 + 218	
425	201/181 · 181/171 · 76/67	181 + 98 + 218	
426	201/191 · 191/171 · 76/67	88 + 191 + 218	
427	256/243 · 76/67 · 5427/4864	90 + 218 + 190	EULER

This complex genus is expanded from number 427, which is called "old chromatic" in Euler's text (Euler [1739] 1960, 177). The tuning is clearly diatonic, however, and must be in error. It may have been intended to represent Boethius's 19/16 (76/64) chromatic. The apyknon is 67/57 (280 cents).

D11. Characteristic Interval 17/15 217 cents

428	40/37 · 37/34 · 17/15	135 + 146 + 217	
429	10/9 · 18/17 · 17/15	182 + 99 + 217	KORNERUP
430	20/19 · 19/17 · 17/15	89 + 192 + 217	PTOLEMY
431	15/14 · 56/51 · 17/15	119 + 162 + 217	
432	80/77 · 77/68 · 17/15	66 + 215 + 217	
433	12/11 · 55/51 · 17/15	151 + 131 + 217	
434	120/109 · 109/102 · 17/15	166 + 115 + 217	
435	120/113 · 113/102 · 17/15	104 + 177 + 217	
436	24/23 · 115/102 · 17/15	74 + 208 + 217	
437	160/153 · 9/8 · 17/15	77 + 204 + 217	

This genus divides the 20/17 (281 cents). Number 429 is Kornerup's (1934,

10) Lydian. Genus number 430 is Ptolemy's interpretation of Aristoxenos's intense diatonic, 6 + 12 + 12 "parts" (Wallis 1682, 172). Kornerup refers to it as Dorian. Number 432 is a hypothetical Ptolemaic interpretation of 4.5 + 13.5 + 12 "parts", a mixed chromatic and diatonic genus not in Ptolemy. Number 437 generates the 34:40:51 triad and tritriadic. The remaining divisions are experimental neo-Aristoxenian genera with a constant upper interval of 12 "parts."

D12. CHARACTERISTIC INTERVAL 112/99 214 CENTS

438	66/61 · 61/56 · 112/99	136 + 148 + 214	
439	99/94 · 47/42 · 112/99	90 + 195 + 214	
440	99/89 · 89/84 · 112/99	184 + 100 + 214	
441	10/9 · 297/280 · 112/99	182 + 102 + 214	
442	22/21 · 9/8 · 112/99	81 + 204 + 214	

This very complex genus divides the 33/28 (284 cents). Number 442 generates the 22:28:33 tritriadic and its conjugate.

D13. CHARACTERISTIC INTERVAL 44/39 209 CENTS

443	12/11 · 13/12 · 44/39	151 + 139 + 209	YOUNG
444	39/35 · 35/33 · 44/39	187 + 102 + 209	
445	39/37 · 37/33 · 44/39	91 + 198 + 209	
446	44/39 · 9/8 · 104/99	209 + 204 + 85	

The first division is William Lyman Young's "exquisite 3/4-tone Hellenic lyre" (Young 1961, 5). The apyknon is 13/11 (289 cents). Number 446 generates the 22:26:33 tritriadic scale.

D14. CHARACTERISTIC INTERVAL 152/135 205 CENTS

447	90/83 · 83/76 · 152/135	140 + 153 + 205	
448	135/128 · 64/57 · 152/135	92 + 201 + 205	
449	135/121 · 121/114 · 152/135	190 + 103 + 205	
450	20/19 · 9/8 · 152/135	89 + 204 + 205	

This genus derives from the 30:38:45 triad and divides its upper interval, 45/38 (293 cents). Number 450 generates the 30:38:45 tritriadic and its conjugate.

D15. CHARACTERISTIC INTERVAL 9/8 204 CENTS

451	64/59 · 59/54 · 9/8	141 + 153 + 204	SAFIYU-D-DIN
452	48/43 · 86/81 · 9/8	190 + 104 + 204	SAFIYU-D-DIN
453	96/91 · 91/81 · 9/8	93 + 202 + 204	
454	256/243 · 9/8 · 9/8	90 + 204 + 204	PYTHAGORAS?

455	16/15 · 9/8 · 10/9	112 + 204 + 182	PTOLEMY, DIDYMOS
456	2187/2048 · 65536/59049 · 9/8	114 + 180 + 204	ANONYMOUS
457	9/8 · 12/11 · 88/81	204 + 151 + 143	AVICENNA
458	13/12 · 9/8 · 128/117	139 + 204 + 156	AVICENNA
459	14/13 · 9/8 · 208/189	128 + 204 + 166	AVICENNA
460	9/8 · 11/10 · 320/297	204 + 165 + 129	AL-FARABI
461	9/8 · 15/14 · 448/405	204 + 119 + 175	
462	9/8 · 17/16 · 512/459	204 + 105 + 189	
463	9/8 · 18/17 · 272/243	204 + 99 + 195	
464	9/8 · 19/18 · 64/57	204 + 94 + 201	
465	56/51 · 9/8 · 68/63	162 + 204 + 132	
466	9/8 · 200/189 · 28/25	204 + 98 + 196	
467	184/171 · 9/8 · 76/69	127 + 204 + 167	
468	32/29 · 9/8 · 29/27	170 + 204 + 124	
469	121/108 · 9/8 · 128/121	197 + 204 + 97	PARTCH
470	9/8 · 4096/3645 · 135/128	204 + 202 + 92	
471	9/8 · 7168/6561 · 243/224	204 + 153 + 141	
472	35/32 · 1024/945 · 9/8	204 + 139 + 204	

The apyknon of this genus is 32/27 (294 cents). Numbers 451 and 452 are Safiyu-d-Din's weak and strong forms of the division, respectively. The attribution of the tetrachord number 454 to Pythagoras is questionable, though traditional—the diatonic scale in "Pythagorean" intonation antedates him by a millennium or so in the Near East (Duchesne-Guillemin 1963, 1969). The earliest reference to this scale in a European language is in Plato's *Timaeus.* Number 455 is attributed to both Ptolemy and Didymos because their historically important definitions differed in the order of the intervals. Ptolemy's is the order shown; Didymos placed the 9/8 at the top. Ptolemy's order generates the major mode in just intonation. Its retrograde, 10/9 · 9/8 · 16/15, yields the natural minor and new scale of Redfield (1928). Number 456 is a "Pythagorean" form extracted from the anonymous treatise in D'Erlanger (1939). In reverse order, it appears in the Turkish scales of Palmer (1967?). Numbers 457–460 are also from D'Erlanger. Numbers 457 and 458 generate the 18:22:27 and 26:32:39 tritriadics and their conjugates. These and the tetrachord from Al-Farabi, number 459, resemble modern Islamic tunings (Sachs 1943, 283). Numbers 464 and 465 generate the 16:19:24 and the 14:17:21 tritriadics. In theory, any tetrachord containing a 9/8 generates a tritriadic and its conjugate, but in practice the majority

are not very consonant. Examples are numbers 467 and 468 which generate the 38:46:57 and 24:29:36 tritriadics with mediants of 23/19 and 29/24. Number 469 is an adventitious tetrachord from Partch (1974, 165). Numbers 470–472 are from chapter 4. The last two resemble some of the Islamic tunings of the Middle Ages. The remaining tunings are proposed approximations to Islamic or syntonic diatonic tetrachords.

D16. Characteristic Interval 160/143 194 cents

473	11/10 · 13/12 · 160/143	165 + 139 + 194	AL-FARABI

This tetrachord is from Al-Farabi (D'Erlanger 1930, 112). It did not seem worthwhile to explore this genus further because the ratios would be complex and often larger than 160/143 itself.

D17. Characteristic Interval 10/9 182 cents

474	12/11 · 11/10 · 10/9	151 + 165 + 182	PTOLEMY
475	10/9 · 10/9 · 27/25	182 + 182 + 133	AL-FARABI
476	10/9 · 13/12 · 72/65	182 + 139 + 177	AVICENNA

The apyknon is 6/5 and the majority of potential divisions have intervals larger than the 10/9. Number 474 is Ptolemy's homalon or equable diatonic, a scale which has puzzled theorists, but which seems closely related to extant tunings in the Near East. Ptolemy described it as sounding rather foreign and rustic. Could he have heard it or something similar and written it down in the simplest ratios available? It certainly sounds fine, perhaps a bit like 7-tone equal temperament with perfect fourths and fifths. The Avicenna and Al-Farabi references are from D'Erlanger (1935), and Ptolemy (Wallis 1682).

Reduplicated tetrachords

These genera are arranged by the reduplicated interval in descending order of size.

477	11/10 · 11/10 · 400/363	165 + 165 + 168		R1
478	12/11 · 12/11 · 121/108	151 + 151 + 197	AVICENNA	R2
479	13/12 · 13/12 · 192/169	139 + 139 + 221	AVICENNA	R3
480	14/13 · 14/13 · 169/147	128 + 128 + 241	AVICENNA	R4
481	15/14 · 15/14 · 784/675	119 + 119 + 259	AVICENNA	R5
482	2187/2048 · 16777216/14348907 · 2187/2048	114 + 271 + 114	PALMER	R6
483	17/16 · 17/16 · 1024/867	105 + 105 + 288		R7
484	18/17 · 18/17 · 289/243	99 + 99 + 300		R8
485	256/243 · 256/243 · 19688/16384	90 + 90 + 318		R9
486	22/21 · 147/121 · 22/21	81 + 337 + 81		R10

487	25/24 · 25/24 · 768/625	71 + 71 + 357	RI1
488	28/27 · 28/27 · 243/196	63 + 63 + 372	RI2
489	34/33 · 34/33 · 363/289	52 + 52 + 395	RI3
490	36/35 · 36/25 · 1225/972	49 + 49 + 401	RI4
491	40/39 · 40/39 · 507/400	44 + 44 + 410	RI5
492	46/45 · 46/45 · 675/529	38 + 38 + 422	RI6

While a number of other small intervals could be used to construct analogous genera, the ones given here seem the most important and most interesting. Number 477 is an approximation in just intonation to the equally tempered division of the 4/3. See number 722 for the semi-tempered version. The Avicenna genera are from vol. 2, pages 122–123 and page 252 of D'Erlanger. The Palmer genus is from his booklet on Turkish music (1967?). This genus is very close to Helmholtz's chromatic 16/15 · 75/64 · 16/15. The 18/17 genus is also nearly equally tempered and is inspired by Vincenzo Galilei's lute fretting (Barbour 1951, 57). Number 486 is nearly equal to 1/1 $\pi/3$ $4/\pi$ 4/3, a theoretical genus using intervals of 11 to approximate intervals of π. Numbers 487 and 488 come from Winnington-Ingram's (1932) suggestion that Aristoxenos's soft and hemiolic chromatics were somewhat factitious genera resulting from the duplication of small, but known, intervals. The remaining tetrachords are in the spirit of Avicenna and Al-Farabi.

Miscellaneous tetrachords

The tetrachords in this section are those that were discovered in the course of various theoretical studies but which were not judged to be of sufficient interest to enter in the Main Catalog. Many of these genera have unusual CIs which were not thought worthy of further study. The fourth and fifth columns give the ratio of the pyknon or apyknon and its value in cents.

493	176/175 · 175/174 · 29/22	10 + 10 + 478	88/87	20	M1
494	25/19 · 931/925 · 148/147	475 + 11 + 12	76/75	23	M2

This tetrachord is generated by the second of the summation procedures of chapter 5.

495	128/127 · 127/126 · 21/16	14 + 14 + 471	64/63	27	M3
496	21/16 · 656/651 · 124/123	471 + 13 + 14	64/63	27	M4

Another summation tetrachord from chapter 4.

497	104/103 · 103/102 · 17/13	17 + 17 + 464	52/51	34	M5
498	17/13 · 429/425 · 100/99	464 + 16 + 17	52/51	34	M6

Another summation tetrachord from chapter 4.

499	98/97 · 97/96 · 64/49	18 + 18 + 462	49/48	36	M7
500	92/91 · 91/90 · 30/23	19 + 19 + 460	46/45	38	M8
501	90/89 · 89/88 · 176/135	19 + 20 + 459	45/44	39	M9
502	88/87 · 87/86 · 43/33	20 + 20 + 458	44/43	40	M10
503	86/85 · 85/84 · 56/43	20 + 20 + 457	43/42	41	M11
504	84/83 · 83/82 · 82/63	21 + 21 + 456	42/41	42	M12
505	82/81 · 81/80 · 160/123	21 + 22 + 455	41/40	43	M13

These genera contain intervals which are probably too small for use in most music. However, Harry Partch and Julián Carrillo, among others, have used intervals in this range.

506	13/10 · 250/247 · 76/74	454 + 21 + 23	40/39	44	M14

Another summation tetrachord from chapter 4.

507	78/77 · 77/76 · 152/117	22 + 23 + 453	39/38	45	M15
508	76/75 · 76/75 · 74/57	23 + 23 + 452	38/37	46	M16
509	74/73 · 73/72 · 48/31	24 + 24 + 451	37/36	47	M17
510	70/69 · 69/68 · 136/105	25 + 25 + 448	35/34	50	M18
511	22/17 · 357/352 · 64/63	446 + 24 + 27	34/33	52	M19

Another summation tetrachord from chapter 4.

512	58/57 · 57/56 · 112/87	30 + 31 + 437	29/28	61	M20
513	87/80 · 43/42 · 112/87	20 + 41 + 437	29/28	61	M21
514	87/85 · 85/84 · 112/87	40 + 20 + 437	29/28	61	M22

The preceding are a set of hyperenharmonic genera which divide the dieses between 40/39 and 28/27. Similar but simpler genera will be found in the Main Catalog. Small intervals in this range are clearly perceptible, but have been rejected by most theoreticians, ancient and modern.

515	68/53 · 53/52 · 52/51	431 + 33 + 34	53/51	67	M23
516	136/133 · 133/130 · 65/51	34 + 34 + 420	68/65	78	M24
517	68/67 · 67/65 · 65/51	26 + 52 + 420	68/65	78	M25
518	34/33 · 66/65 · 65/51	52 + 26 + 420	68/65	78	M26
519	68/67 · 67/54 · 18/17	26 + 373 + 99	72/76	125	M27
520	25/24 · 32/31 · 31/25	71 + 55 + 372	100/93	126	M28
521	68/55 · 55/54 · 18/17	367 + 32 + 99	55/51	131	M29
522	68/67 · 67/63 · 21/17	26 + 107 + 366	68/63	132	M30
523	68/65 · 65/63 · 21/17	78 + 54 + 366	68/63	132	M31
524	36/35 · 256/243 · 315/256	49 + 90 + 359	1024/945	139	M32
525	64/63 · 16/15 · 315/256	27 + 112 + 359	1024/945	139	M33

Numbers 524 and 525 are from Vogel's PIS tuning of chapter 6.

526	64/63 · 2187/2048 · 896/729	27 + 114 + 357	243/224	141	M34
527	36/35 · 135/128 · 896/729	49 + 92 + 357	243/224	141	M35
	This tuning is a close approximation to one produced by the eighth mean (Heath 1921) of chapter 4. It also occurs in Erickson's analysis of Archytas's system and in Vogel's tuning (chapter 6 and Vogel 1963, 197).				
528	28/27·2187/1792·256/243	63 + 345 + 90	7168/6561	153	M36
	This tetrachord appears in Erickson's commentary on Archytas's system with trite synemmenon (112/81, B♮-) added.				
529	16/15·2240/2187·2187/1792	112 + 41 + 345	7168/6561	153	M37
530	28/27·128/105·135/128	63 + 343 + 92	35/32	141	M38
	Numbers 528–530 are from Vogel's PIS tuning of chapter 6.				
531	17/16·32/31·62/51	105 + 55 + 338	34/31	160	M39
532	20/19·57/47·47/45	89 + 334 + 75	188/171	164	M40
	Number 532 is a possible Byzantine chromatic.				
533	360/349·349/327·109/90	54 + 113 + 332	120/109	166	M41
534	24/23·115/109·109/90	74 + 94 + 332	120/109	166	M42
	Number 534 is a hypothetical Ptolemaic interpretation of 5 + 6 + 19 "parts", after Macran (1902).				
535	240/229 · 229/218 · 109/90	81 + 85 + 332	120/109	166	M43
536	19/18 · 24/23 · 23/19	94 + 74 + 330	76/69	167	M44
537	15/14 · 36/35 · 98/81	119 + 49 + 330	54/49	168	M45
	Number 537 occurs in Other Music's gamelan tuning (Henry S. Rosenthal, personal communication).				
538	28/27 · 16/15 · 135/112	63 + 112 + 323	448/405	175	M46
539	24/23 · 115/96 · 16/15	74 + 313 + 112	128/115	185	M47
	A Ptolemaic interpretation of Xenakis's 5 + 19 + 6 "parts" (1971).				
540	256/243 · 243/230 · 115/96	90 + 95 + 313	128/115	185	M48
541	68/67 · 67/56 · 56/51	26 + 310 + 162	224/201	88	M49
542	68/57 · 19/18 · 18/17	305 + 94 + 99	19/17	193	M50
543	15/14 · 266/255 · 68/57	119 + 73 + 305	19/17	193	M51
544	256/243· 243/229 · 229/192	90 + 103 + 305	256/192	193	M52
545	32/31 · 13/12 · 31/26	55 + 139 + 304	104/93	194	M53
546	240/227 · 227/214 · 107/90	96 + 102 + 300	120/107	199	M54
547	360/347 · 347/321 · 107/90	64 + 135 + 300	120/107	199	M55
	This genus is related to (Ps.)-Philolaus's division as 6.5 + 6.5 + 17 "parts". See also chapter 4.				
548	7168/6561 · 36/35 · 1215/1024	153 + 49 + 296	4096/3645	202	M56

549	16/15 · 1215/1024 · 256/243	112 + 296 + 90	4096/3635	202	M57
550	28/27 · 1024/945 · 1215/1024	63 + 139 + 296	4096/3635	202	M58

Numbers 548–550 are from Vogel's PIS tuning of chapter 6.

551	120/113 · 113/106 · 53/45	104 + 111 + 283	60/53	215	M59
552	180/173 · 173/159 · 53/45	69 + 146 + 283	60/53	215	M60
553	90/83 · 166/159 · 53/45	140 + 75 + 283	60/53	215	M61
554	24/23 · 115/106 · 53/45	74 + 141 + 283	60/53	215	M62

Number 554 is a hypothetical Ptolemaic interpretation of 5 + 9 + 16 "parts." The others, numbers 551, 552, and 553 are 1:1, 1:2 and 2:1 divisions of the pyknon.

555	34/29 · 58/57 · 19/17	275 + 30 + 193	58/51	223	M63
556	10/9 · 117/100 · 40/39	182 + 272 + 44	400/351	226	M64
557	120/113 · 113/97 · 97/90	104 + 264 + 130	388/339	234	M65

This genus is a Ptolemaic interpretation of Xenakis's 7 + 16 + 7 "parts."

558	13/12 · 55/52 · 64/55	139 + 97 + 262	55/48	236	M66

This genus is generated by the second ratio mean of chapter 4.

559	68/65 · 65/56 · 56/51	78 + 258 + 162	224/195	240	M67
560	12/11 · 297/256 · 256/243	151 + 257 + 90	1024/891	241	M68
561	28/27 · 81/70 · 10/9	63 + 253 + 182	280/243	245	M69

This tetrachord is also found in Erickson's article on Archytas's system with trite synemmenon (112/81, B♭-) added. It also occurs in Vogel's PIS tuning of chapter 6.

562	81/70 · 2240/2187 · 9/8	253 + 41 + 204	280/243	245	M70
563	81/70 · 256/243 · 35/32	253 + 90 + 155	280/243	245	M71
564	135/128 · 7168/6561 · 81/70	92 + 153 + 253	280/243	245	M72

These three tetrachords are from Vogel's PIS tuning of chapter 6.

565	60/59 · 59/51 · 17/15	29 + 252 + 217	68/59	246	M73
566	40/37 · 37/32 · 16/15	135 + 251 + 112	128/111	247	M74

This is a Ptolemaic interpretation of Athanasopoulos's 9 + 15 + 6 "parts."

567	16/15 · 280/243 · 243/224	112 + 245 + 141	81/70	253	M75
568	36/35 · 9/8 · 280/243	49 + 204 + 245	81/70	253	M76
569	8/7 · 81/80 · 280/243	231 + 22 + 245	81/70	253	M77

These three tetrachords are from Vogel's PIS tuning of chapter 6.

570	46/45 · 132/115 · 25/22	38 + 239 + 221	115/99	259	M78
571	16/15 · 12/11 · 55/48	112 + 151 + 236	64/55	262	M79

This is an approximation to the soft diatonic of Aristoxenos, 1/2 + 3/4 + 1 1/4 tones, 6 + 9 + 15 "parts."

572	10/9 · 63/55 · 22/21	182 + 235 + 81	220/189	263	M80

This is another tetrachord from Partch ([1949] 1974, 165), presented as an approximation to a tetrachord of the "Ptolemaic sequence," or major mode in 5-limit just intonation.

573	30/29 · 116/103 · 103/90	59 + 206 + 234	120/103	264	M81
574	360/343 · 343/309 · 103/90	84 + 181 + 234	120/103	264	M82
575	40/39 · 143/125 · 25/22	44 + 233 + 221	500/429	265	M83
576	68/65 · 65/57 · 19/17	78 + 227 + 193	76/65	271	M84
577	256/243 · 729/640 · 10/9	90 + 225 + 182	2560/2187	273	M85
578	30/29 · 58/51 · 17/15	59 + 223 + 217	34/29	275	M86
579	23/21 · 14/13 · 26/23	158 + 128 + 212	46/39	286	M87
580	23/22 · 44/39 · 26/23	77 + 209 + 212	46/39	286	M88
581	14/13 · 260/231 · 11/10	128 + 205 + 165	77/65	293	M89
582	4096/3645 · 35/32 · 243/224	202 + 155 + 141	1215/1024	296	M90

From Vogel's PIS tuning of chapter 6.

583	38/35 · 35/32 · 64/57	142 + 155 + 201	19/16	298	M91
584	19/17 · 17/16 · 64/57	193 + 105 + 201	19/16	298	M92
585	11/10 · 95/88 · 64/57	165 + 135 + 201	19/16	298	M93

The apyknon of genera numbers 583–585 is 19/16. The 1:2 division is listed as D15 (9/8), number 464.

586	240/221 · 221/202 · 101/90	143 + 156 + 200	120/101	298	M94
587	15/14 · 112/101 · 101/90	119 + 179 + 200	120/101	298	M95
588	120/113 · 113/101 · 101/90	104 + 194 + 200	120/101	298	M96
589	533/483 · 575/533 · 28/25	171 + 131 + 196	25/21	302	M97

A mean tetrachord of the first kind from chapter 4.

590	19/17 · 85/76 · 16/15	193 + 194 + 112	304/255	304	M98
591	19/17 · 1156/1083 · 19/17	193 + 113 + 193	68/57	305	M99

Two tetrachords from Thomas Smith (personal communication, 1989).

592	68/63 · 21/19 · 19/17	132 + 173 + 193	68/57	305	M100
593	10/9 · 108/97 · 97/90	182 + 186 + 130	97/90	368	M101

Tetrachords in equal temperament

The tetrachords listed in this section of the Catalog are the genera of Aristoxenos and other writers in this tradition (chapter 3). Included also are those genera which appear as vertices in the computations of Rothenberg's propriety function and other descriptors, and various neo-Aristoxenian genera. These are all divisions of the tempered fourth (500 cents).

The "parts" of the fourth used to describe the scales of Aristoxenos are, in fact, the invention of Cleonides, a later Greek writer, as Aristoxenos spoke only of fractional tones. The invention has proved both useful and durable, for not only the later classical writers, but also the Islamic theorists and the modern Greek Orthodox church employ the system, though the former have often doubled the number to avoid fractional parts in the hemiolic chromatic and a few other genera.

Until recently, the Greek church has used a system of 28 parts to the fourth (Tiby 1938), yielding a theoretical octave of 68 (28 + 12 + 28) tones rather than the 72 (30 + 12 + 30 = 72) or 144 (60 + 24 + 60 = 144 in the hemiolic chromatic and rejected genera) of the Aristoxenians. The 68-tone equal temperament has a fourth of only 494 cents.

Note that a number of the Orthodox liturgical tetrachords are meant to be permuted in the formation of the different modes (echoi). This operation may be applied to the historical and neo-Aristoxenian ones as well.

ARISTOXENIAN STYLE TETRACHORDS

594	2 + 2 + 26	33 + 33 + 433	CHAPTER 4	T1
595	2.5 + 2.5 + 25	42 + 42 + 417	CHAPTER 4	T2
596	2 + 3 + 25	33 + 50 + 417	CHAPTER 4	T3
597	3 + 3 + 24	50 + 50 + 400	ARISTOXENOS	T4
598	2 + 4 + 24	33 + 67 + 400	CHAPTER 4	T5
599	2 + 5 + 23	33 + 83 + 383	CHAPTER 4	T6
600	7/3 + 14/3 + 23	39 + 78 + 383	CHAPTER 4	T7
601	4 + 3 + 23	67 + 50 + 383	CHAPTER 3	T8
602	3.5 + 3.5 + 23	58 + 58 + 383	CHAPTER 4	T9
603	2 + 6 + 22	33 + 100 + 367	CHAPTER 4	T10
604	4 + 4 + 22	66 + 66 + 367	ARISTOXENOS	T11
605	8/3 + 16/3 + 22	44 + 89 + 367	CHAPTER 4	T12
606	3 + 5 + 22	50 + 83 + 367	CHAPTER 4	T13
607	4.5 + 3.5 + 22	75 + 58 + 367	ARISTOXENOS	T14
608	2 + 7 + 21	33 + 117 + 350	CHAPTER 4	T15
609	3 + 6 + 21	50 + 100 + 350	CHAPTER 4	T16
610	4.5 + 4.5 + 21	75 + 75 + 350	ARISTOXENOS	T17
611	4 + 5 + 21	67 + 83 + 350	CHAPTER 4	T18
612	6 + 3 + 21	100 + 50 + 350	ARISTOXENOS	T19
613	6 + 20 + 4	100 + 333 + 67	SAVAS	T20
614	10/3 + 20/3 + 20	56 + 111 + 333	CHAPTER 4	T21

615	5 + 5 + 20	83 + 83 + 334	CHAPTER 4	T22
616	5.5 + 5.5 + 19	92 + 92 + 317	CHAPTER 4	T23
617	11/3 + 22/3 + 19	61 + 122 + 317	CHAPTER 4	T24
618	5 + 19 + 6	83 + 317 + 100	XENAKIS	T25
619	5 + 6 + 19	83 + 100 + 317	MACRAN	T26
620	2 + 10 + 18	33 + 167 + 300	CHAPTER 4	T27
621	3 + 9 + 18	50 + 150 + 300	CHAPTER 4	T28
622	4 + 8 + 18	67 + 133 + 300	ARISTOXENOS	T29
623	4.5 + 7.5 + 18	75 + 125 + 300	CHAPTER 4	T30
624	6 + 6 + 18	100 + 100 + 300	ARISTOXENOS	T31
625	5 + 7 + 18	83 + 117 + 300	CHAPTER 4	T32
626	6 + 18 + 6	100 + 300 + 100	ATHANASOPOULOS	T33
627	13/3 + 26/3 + 17	72 + 144 + 283	CHAPTER 4	T34
628	6.5 + 6.5 + 17	108 + 108 + 283	CHAPTER 4	T35
629	2 + 16 + 12	33 + 267 + 200	CHAPTER 4	T36
630	14/3 + 28/3 + 16	78 + 156 + 267	CHAPTER 4	T37
631	5 + 9 + 16	83 + 150 + 267	WINNINGTON-INGRAM	T38
632	8 + 16 + 6	133 + 267 + 100	SAVAS	T39
633	7 + 16 + 7	117 + 267 + 117	XENAKIS; CHAP. 4	T40
634	2 + 13 + 15	33 + 217 + 250	CHAPTER 4	T41
635	3 + 12 + 15	50 + 200 + 250	CHAPTER 4	T42
636	4 + 11 + 15	67 + 183 + 250	CHAPTER 4	T43
637	5 + 10 + 15	83 + 167 + 250	CHAPTER 4	T44
638	6 + 9 + 15	100 + 150 + 250	ARISTOXENOS	T45
639	7 + 8 + 15	117 + 133 + 250	CHAPTER 4	T46
640	7.5 + 7.5 + 15	125 + 125 + 250	CHAPTER 4	T47
641	9 + 15 + 6	150 + 250 + 100	ATHANASOPOULOS	T48
642	2 + 14 + 14	33 + 233 + 233	CHAPTER 4	T49
643	4 + 14 + 12	67 + 233 + 200	ARISTOXENOS	T50
644	5 + 11 + 14	83 + 183 + 233	WINNINGTON-INGRAM	T51
645	16/3 + 32/3 + 14	89 + 178 + 233	CHAPTER 4	T52
646	8 + 8 + 14	133 + 133 + 233	CHAPTER 4	T53
647	4.5 + 13.5 + 12	75 + 225 + 200	ARISTOXENOS	T54
648	5 + 12 + 13	83 + 200 + 217	CHAPTER 4	T55
649	4 + 13 + 13	67 + 217 + 217	CHAPTER 4	T56
650	17/3 + 34/3 + 13	94 + 189 + 217	CHAPTER 4	T57
651	8.5 + 8.5 + 13	142 + 142 + 217	CHAPTER 4	T58

652 6 + 12 + 12 — 100 + 200 + 200 — ARISTOXENOS — T59

Savas, Xenakis and Athanasopoulos all give permutations of this tetrachord in their lists of Orthodox church forms.

653 12 + 11 + 7 — 200 + 183 + 117 — XENAKIS — T60

Xenakis (1971) permits several permutations of this approximation to Ptolemy's intense diatonic.

654 10 + 8 + 12 — 167 + 133 + 200 — SAVAS — T61

The form 8 + 12 + 10 is Savas's "Barys diatonic" (Savas 1965).

655 12 + 9 + 9 — 200 + 150 + 150 — AL-FARABI; CH. 4 — T62

656 8 + 11 + 11 — 133 + 183 + 183 — CHAPTER 4 — T63

This tuning is close to 27/25 · 10/9 · 10/9.

657 9.5 + 9.5 + 11 — 158 + 158 + 183 — CHAPTER 4 — T64

658 10 + 10 + 10 — 166 + 167 + 167 — AL-FARABI — T65

Tiby's Greek Orthodox tetrachords of 28 parts to the fourth of 494 cents.

659 12 + 13 + 3 — 212 + 229 + 53 — TIBY — T66

660 12 + 5 + 11 — 212 + 88 + 194 — TIBY — T67

661 12 + 9 + 7 — 212 + 159 + 124 — TIBY — T68

662 9 + 12 + 7 — 159 + 212 + 124 — TIBY — T69

See Tiby (1938) for numbers 659–662.

TEMPERED TETRACHORDS IN CENTS

663 22.7 + 22.7 + 454.5 — CHAPTER 5 — T70

664 37.5 + 37.5 + 425 — CHAPTER 5 — T71

665 62.5 + 62.5 + 375 — CHAPTER 5 — T72

Tetrachord numbers 663– 665 are categorical limits in the classification scheme of 5-9.

666 95 + 115 + 290 — T73

This tetrachord was designed to fill a small gap in tetrachordal space. See 9-4, 9-5, and 9-6.

667 89 + 289 + 122 — CHAPTER 5 — T74

668 87.5 + 287.5 + 125 — CHAPTER 5 — T75

669 83.3 + 283.3 + 133.3 — CHAPTER 5 — T76

670 75 + 275 + 150 — CHAPTER 5 — T77

671 100 + 275 + 125 — CHAPTER 5 — T78

672 55 + 170 + 275 — T79

This tetrachord was designed to fill a small gap in tetrachordal space.

673 66.7 + 266.7 + 166.7 — CHAPTER 5 — T80

674 233.3 + 16.7 + 250 — CHAPTER 5 — T81

675 225 + 25 + 250 CHAPTER 5 T82

676 66.7 + 183.3 + 250 CHAPTER 5 T83

677 75 + 175 + 250 CHAPTER 5 T84

678 125 + 125 + 250 CHAPTER 5 T85

679 105 + 145 + 250 T86

680 110 + 140 + 250 T87

Tetrachord numbers 679 and 680 fill possible gaps in tetrachordal space.

681 87.5 + 237.5 + 175 CHAPTER 5 T88

682 233.3 + 166.7 + 100 CHAPTER 5 T89

683 212.5 + 62.5 + 225 CHAPTER 5 T90

684 225 + 75 + 200 CHAPTER 5 T91

685 225 + 175 + 100 CHAPTER 5 T92

686 87.5 + 187.5 + 225 CHAPTER 5 T93

687 212.5 + 162.5 + 125 CHAPTER 5 T94

688 100 + 187.5 + 212.5 CHAPTER 5 T95

689 212.5 + 137.5 + 150 CHAPTER 5 T96

690 200 + 125 + 175 CHAPTER 5 T97

691 145 + 165 + 190 T98

This tetrachord was designed to fill a small gap in tetrachordal space.

Semi-tempered tetrachords

The tetrachords in this section contain both just and tempered intervals. Two of these genera are literal interpretations of late Classical tuning theory. A number are based on the assumption that Aristoxenos intended to divide the perfect fourth (4/3), a rather doubtful hypothesis. The remainder are mean tetrachords from chapter 4 with medial 9/8. Formally, these latter tetrachords are generators of tritriadic scales. In all cases they span a pure 4/3.

692 $16/(9\sqrt{3}) \cdot 16/(9\sqrt{3}) \cdot 81/64$ 45 + 45 + 408 S1

Number 692 is Barbera's (1978) literal interpretation of Nicomachos's enharmonic as 1/2 semitone + 1/2 semitone + ditone, where the 1/2 semitone is the square root of 256/243, also written as $16 \cdot \sqrt{3} / 27$.

693 $1.26376 \cdot 1.05321 \cdot 1.00260$ 405 + 88 + 4 S2

This mean tetrachord of the second kind is generated by mean 9.

694 $(4/3)^{1/10} \cdot (4/3)^{1/10} \cdot (4/3)^{8/10}$ 50 + 50 + 398 S3

This tetrachord is a literal interpretation of Aristoxenos's enharmonic under Barbera's (1978) assumption that Aristoxenos's meant the perfect fourth 4/3. In Cleonides's cipher, it is 3 + 3 + 24 parts.

695 $(4/3)^{2/15} \cdot (4/3)^{2/15} \cdot (4/3)^{11/15}$ 66 + 66 + 365 S4

This tetrachord is a semi-tempered interpretation of Aristoxenos's soft chromatic. In Cleonides's cipher, it is 4 + 4 + 22 parts.

696 $(4/3)^{3/20} \cdot (4/3)^{7/60} \cdot (4/3)^{11/15}$ 75 + 58 + 365 S5

This tetrachord is a semi-tempered interpretation of a genus rejected by Aristoxenos. It somewhat resembles Archytas's enharmonic. In Cleonides's cipher, it is 4.5 + 3.5 + 22 parts.

697 $(4/3)^{3/20} \cdot (4/3)^{3/20} \cdot (4/3)^{7/10}$ 75 + 75 + 349 S6

This tetrachord is a semi-tempered interpretation of Aristoxenos's hemiolic chromatic. In Cleonides's cipher, it is 4.5 + 4.5 + 21 parts.

698 $(4/3)^{1/5} \cdot (4/3)^{1/10} \cdot (4/3)^{7/10}$ 100 + 50 + 349 S7

This tetrachord is a semi-tempered interpretation of a genus rejected by Aristoxenos. In Cleonides's cipher, it is 6 + 3 + 21 parts.

699 1.21677 · 1.03862 · 1.05505 340 + 66 + 93 S8

This mean tetrachord of the first kind is generated by mean 9.

700 $(4/3)^{1/5} \cdot (4/3)^{1/5} \cdot (4/3)^{3/5}$ 100 + 100 + 299 S9

This tetrachord is a semi-tempered interpretation of Aristoxenos's intense chromatic. In Cleonides's cipher, it is 6 + 6 + 18 parts.

701 $(4/3)^{2/15} \cdot (4/3)^{4/15} \cdot (4/3)^{3/5}$ 66 + 133 + 299 S10

This tetrachord is a semi-tempered interpretation of a genus rejected by Aristoxenos. It closely resembles Archytas's chromatic In Cleonides's cipher, it is 4 + 8 + 18 parts.

702 $3\sqrt{2}/4 \cdot 3\sqrt{2}/4 \cdot 32/27$ 102 + 102 + 294 S11

This tetrachord is implied by writers such as Thrasyllus who did not give numbers for the chromatic, but stated only that it contained a 32/27 and a 1:1 pyknon (Barbera 1978). The semitones are the square root of 9/8.

703 1.18046 · 1.06685 · 1.05873 287 + 112 + 99 S12

This mean tetrachord of the second kind is generated by mean 5.

704 1.05956 · 1.06763 · 1.17876 100 + 113 + 285 S13

This mean tetrachord of the first kind is generated by mean 13.

705 1.17867 · 1.06763 · 1.05956 285 + 113 + 100 S14

This mean tetrachord of the second kind is generated by mean 14.

706 1.17851 · 1.06771 · 1.05963 284 + 113 + 100 S15

This mean tetrachord of the second kind is generated by mean 17.

707 1.17851 · 1.06771 · 1.05963 282 + 114 + 101 S16

This mean tetrachord of the second kind is generated by mean 6.

708 $(4/3)^{1/5} \cdot (4/3)^{3/10} \cdot (4/3)^{1/2}$ 100 + 149 + 250 S17
This tetrachord is a semi-tempered interpretation of Aristoxenos's soft diatonic. In Cleonides's cipher, it is 6 + 9 + 15 parts.

709 1.07457 · 1.07457 · 1.154701 125 + 125 + 249 S18
This mean tetrachord of the first kind is generated by mean 2. The corresponding tetrachord of the second kind has the same intervals in reverse order.

710 $(4/3)^{2/15} \cdot (4/3)^{7/15} \cdot (4/3)^{2/5}$ 66 + 232 + 199 S19
This tetrachord is a semi-tempered interpretation of Aristoxenos's diatonic with soft chromatic diesis. In Cleonides's cipher, it is 4 + 14 + 12 parts.

711 1.13847 · 1.1250 · 1.0410 225 + 204 + 70 S20
This mean tetrachord of the third kind is produced by mean 5.

712 $(4/3)^{3/20} \cdot (4/3)^{9/20} \cdot (4/3)^{2/5}$ 75 + 224 +199 S21
This tetrachord is a semi-tempered interpretation of Aristoxenos's diatonic with hemiolic chromatic diesis. In Cleonides's cipher, it is 4.5 +13.5 + 12 parts.

713 1.13371 · 1.1250 · 1.04540 217 + 204 + 77 S22
This mean tetrachord of the third kind is produced by mean 14. In reverse order, it is generated by mean 13.

714 1.13315 · 1.1250 · 1.04595 216 + 204 + 78 S23
This mean tetrachord of the third kind is produced by the root mean square mean 17.

715 1.09185 · 1.07803 · 1.13278 152 + 130 + 216 S24
This mean tetrachord of the first kind is produced by mean 6.

716 1.09291 · 1.0783328 · 1.13137 154 + 131 + 214 S25
This mean tetrachord of the first kind is produced by mean 17.

717 1.09301 · 1.07837 · 1.13122 154 + 131 + 213 S26
This mean tetrachord of the first kind is produced by mean 14. In reverse order is the tetrachord of the second kind generated by mean 13.

718 1.09429 · 1.07874 · 1.12950 156 + 131 + 211 S27
This mean tetrachord of the first kind is produced by mean 5.

719 1.12950 · 1.1250 · 1.04930 211 + 204 + 83 S28
This mean tetrachord of the third kind is produced by mean 6.

720 1.08866 · 1.1250 · 1.08866 147 + 204 + 147 S29
This mean tetrachord of the third kind is produced by the second or geometric mean.

721 $(4/3)^{1/5} \cdot (4/3)^{2/5} \cdot (4/3)^{2/5}$ 100 + 199 + 199 S30
This tetrachord is a semi-tempered interpretation of Aristoxenos's intense diatonic. In Cleonides's cipher, it is 6 + 12 + 12 parts.

722 $(4/3)^{1/3} \cdot (4/3)^{1/3} \cdot (4/3)^{1/3}$ 166 + 166 + 166 S31
Number 722 is the equally tempered division of the 4/3 into three parts. It is the semi-tempered form of Ptolemy's equable diatonic and of the Islamic neo-Aristoxenian approximation 10 + 10 + 10.

723 $(4/3)^{2/5} \cdot (4/3)^{3/10} \cdot (4/3)^{3/10}$ 200 + 149 + 149 S32
Number 723 is the semi-tempered version of the Islamic neo-Aristoxenian genus 12 + 9 + 9 parts.

Source index

The sources of the tetrachords listed below are the discoverers, when known, or the earliest reference known at the time of writing. Further scholarship may change some of these attributions. Because the Islamic writers invariably incorporated Ptolemy's tables into their compilations, they are credited with only their own tetrachords. The same criterion was applied to other historical works.

Permutations are not attributed separately except in notable cases such as that of Didymus's and Ptolemy's mutual use of forms of 16/15 · 9/8 · 10/9. Doubtful attributions are marked with a question mark.

For more information, including literature citations, one should refer to the entries in the Main Catalog. Uncredited tetrachords are those of the author.

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